Stationary separability of quantum Hamiltonian systems

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Conjecture

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The result

We proved the conjecture for the class of Stäckel systems quadratic in momenta, and separation curve of Laurent polynomial.

Preliminaries

Assumptions

(M, Π)-Poisson manifold ($2n$-dimensional phase space)
(x, p)-local canonical coordinates on M.

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Systems of our interest

Consider Liouville integrable system on M

$$H_r = \frac{1}{2} A_r^{ij} p_i p_j + V_r(x), \quad \{H_r, H_s\}_\Pi = \Pi(dH_r, dH_s) = 0, \quad r, s = 1, ..., n.$$

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"Natural" Riemannian structure

If we define $G = A_1$, then $M = T^*Q$ where (Q, g) is some Riemann space, $g = G^{-1}$ and A_r, $r = 2, ..., n$ are Killing tensors of metric G.

Preliminaries

Admissible quantizations (Błaszak and Domanski 2013)

Admissible quantizations with respect to the metric G is represented by n Hermitean operators

$$\hat{H}_r = -\frac{1}{2}\hbar^2\left(\nabla_i A_r^{ij}\nabla_j + \frac{1}{4}bA_r^{ij}{}_{;ij} + \frac{1}{4}aA_r^{ij}R_{ij}\right) + V_r(x), \quad a, b \in R,$$

from Hilbert space

$$\mathcal{H} = L^2(Q, |\det g|^{\frac{1}{2}} dx), \quad g = G^{-1},$$

where ∇_i - operator of covariant derivative with respect to Levi-Civita connection, $;i$ is the covariant derivative and R_{ij} is a Ricci tensor.

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Minimal quatization (Duval and Valent 2005)

Minimal quantization, given by the choice $a = b = 0$ is represented by n Hermitean operators

$$\hat{H}_r = -\frac{1}{2}\hbar^2\nabla_i A_r^{ij}\nabla_j + V_r(x), \qquad r = 1, ..., n.$$

Quantum separability: old theory

The well established knowledge: Robertson 1927, Eisenhart 1934, Benenti, Chanu and Rastelli 2002

The Schrödinger equations for all Hamiltonians are multiplicative separable in some coordinate system if the following conditions hold:

- the coordinates in question separate the classical Hamiltonians,
- in these coordinates the **Robertson condition** is satisfied:

$$R_{ij} = \frac{3}{2}\partial_i \Gamma_j = 0, \qquad i \neq j;$$

  here R_{ij} is the Ricci tensor and the metrically contracted Christoffel symbols Γ_i are defined by $\Gamma_i = g_{il} G^{jk} \Gamma^l_{jk}$ and in orthogonal coordinates by

$$\Gamma_i = \frac{1}{2}\partial_i \ln \frac{\prod_{k \neq i} G^{kk}}{G^{ii}}.$$

Quantum separability: old theory

The well established knowledge

The Robertson condition follows from the fact that in separation coordinates (λ, μ) Hamiltonian operators take the form

$$\hat{H}_r = -\frac{1}{2}\hbar^2 A_r^{ii}\left(\partial_i^2 - \Gamma_i \partial_i\right) + V_r(\lambda).$$

Commutativity condition, i.a. so called **pre-Robertson condition** (Benenti et al. 2002) takes the form

$$\partial_i^2 \Gamma_j - \Gamma_i \partial_i \Gamma_j = 0, \quad i \neq j.$$

So, if Robertson condition is satisfied, then

$$[\hat{H}_r, \hat{H}_s] = 0, \quad r, s = 1, ..., n.$$

Quantum separability: new results

Adopted Riemannian structure

Let $\bar{G}$ be arbitrary metric for which separation coordinates are orthogonal coordinates. Then

$$H_r = \frac{1}{2}A_r^{ij}p_ip_j + V_r(x) = \frac{1}{2}(T_r\bar{G})_r^{ij}p_ip_j + V_r(x), \qquad r = 1, ..., n$$

where $T_r = A_r\bar{g}$ and $M = T^*Q$, $(Q, \bar{g})$- some Riemann space. Minimal quantization with respect to the metric $\bar{G}$ is represented by n Hermitean operators

$$\hat{\bar{H}}_r = -\frac{1}{2}\hbar^2\bar{\nabla}_iA_r^{ij}\bar{\nabla}_j + V_r(x), \qquad r = 1, ..., n$$

from Hilbert space

$$\mathcal{H} = L^2(Q, |\det\bar{g}|^{\frac{1}{2}}dx)$$

where $\bar{\nabla}_i$ - operator of covariant derivative related to the metric $\bar{g}$.

Quantum separability: new results

Adopted Riemannian structure

In separation coordinates

$$\hat{H}_r = -\frac{1}{2}\hbar^2 A_r^{ii}[\partial_i^2 + (\partial_i \ln T_r^{ii} - \bar{\Gamma}_i)\partial_i] + V_r(\lambda), \qquad r = 1, ..., n \tag{1}$$

and the **generalized Robertson condition** for quantum separability is

$$\partial_j \Theta_i^r \equiv \partial_j(\partial_i \ln T_r^{ii} - \bar{\Gamma}_i) = 0, \quad j \neq i, \quad \Rightarrow \quad \Theta_i^r = \Theta_i^r(\lambda_i). \tag{2}$$

Moreover, the **generalized pre-Robertson condition** is as follows

$$\partial_i^2 \Theta_j^r - \Theta_i^r \partial_i \Theta_j^r = 0, \quad i \neq j, \; r = 1, ..., n,$$

so, again from separability follows quantum Liouville integrability.

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We classified Stäckel systems quadratic in momenta and proved that for any class of such systems, there exists a family of metrices for which we have quantum separability, i.e. conditions (2) are fulfilled and Hamiltonians (1) commute.

Separation coordinates

We consider Liouville-integrable systems having separation relations generated by the following separation curve

$$H_1\lambda^{\gamma_1} + H_2\lambda^{\gamma_2} + \cdots + H_n = \frac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda), \quad i = 1, 2, \ldots, n,$$

where $\gamma_i \in \mathbb{Z}_+$ with normalization $\gamma_n = 0$ and f, σ are arbitrary rational functions.

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The solution has the form

$$H_r = \frac{1}{2}(S_\gamma^{-1})^i_r f(\lambda_i)\mu_i^2 + (S_\gamma^{-1})^i_r \sigma(\lambda_i) = \frac{1}{2}(A_r)^{ii}\mu_i^2 + V_r(\lambda), \quad r = 1, \ldots, n,$$

with Stäckel matrix

$$S_\gamma = \begin{pmatrix} \lambda_1^{\gamma_1} & \lambda_1^{\gamma_2} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ \lambda_n^{\gamma_1} & \lambda_n^{\gamma_2} & \cdots & 1 \end{pmatrix}.$$

Hamiltonians in separation coordinates

Benenti class

$$(\gamma_1, ..., \gamma_n) = (n-1, n-2, ..., 0).$$

Stäckel matrix is a Vandermonde matrix. For "natural" representation

$$A_1^{ii} = G_B^{ii} = \frac{f(\lambda^i)}{\triangle_i}, \quad \triangle_i = \prod_{k \neq i} (\lambda^i - \lambda^k), \quad i = 1, ..., n.$$

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All above metric tensors have a common set of Killing tensors

$$(K_{B_r})^i_i = -\frac{\partial \rho_r}{\partial \lambda^i}, \quad r = 1, ..., n,$$

where $\rho_r(\lambda)$ are Viéte polynomials

$$\rho_1 = -(\lambda^1 + ... + \lambda^n), ..., \rho_n = (-1)^n \lambda^1 \lambda^2 ... \lambda^n.$$

Hamiltonians in separation coordinates

Other classes

For any other class, specified by a particular choice of $\gamma_1, ..., \gamma_n$, the structure of A_r in separation coordinates is as follows

$$A_r = \frac{1}{\rho_\gamma(\lambda)} N_r(K_1, ..., K_n) G_B = (K_r G) = (T_r \overline{G}),$$

where G_B is related metric from Benenti class (with the same f function). $\rho_\gamma(\lambda)$ is known function depending on the choice of $\gamma_1, ..., \gamma_n$ and N_r are polynomial functions of K (Błaszak 2005).

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Quantum Hamiltonians

Minimal quantization in adopted metric $\bar{g}$

$$\hat{H}_r = -\frac{1}{2}\hbar^2 \bar{\nabla}_i A_r^{ij} \bar{\nabla}_j + V_r(x), \qquad r = 1, ..., n$$

Quantum separability: new results

Separability

Application of Stäckel matrix S to the system of eigenvalue problems

$$S\begin{pmatrix}\hat{\hat{H}}_1\Psi\\ \vdots\\ \hat{\hat{H}}_n\Psi\end{pmatrix} = S\begin{pmatrix}E_1\Psi\\ \vdots\\ E_n\Psi\end{pmatrix}$$

separates onto n one-dimensional eigenvalue problems

$$(E_1\lambda_i^{\gamma_1}+E_2\lambda_i^{\gamma_2}+\cdots+E_n)\psi_i = -\frac{1}{2}\hbar^2 f(\lambda_i)\left[\frac{\mathrm{d}^2\psi_i}{\mathrm{d}\lambda_i^2}+\Theta^i(\lambda_i)\frac{\mathrm{d}\psi_i}{\mathrm{d}\lambda_i}\right]+\sigma(\lambda_i)\psi_i,$$

where $\Psi(\lambda_1,\ldots,\lambda_n)=\prod_{i=1}^n\psi_i(\lambda_i)$, under condition

$$\Theta_r^i(\lambda_1,\ldots,\lambda_n)=\Theta^i(\lambda_i).$$

Quantum separability: new results

In order to search for quantum separability let us start from "natural" choice (Robertson, Eisenhart, Benenti et al.) of metric tensor (and hence a Hilbert space): $A_r = K_r G$, $K_1 = I$, $A_1 = G$, where K_r are Killing tensors of G. Because in separation coordinates

$$\partial_i (K_r)^i_i = 0 \Rightarrow \Theta^i_r = \partial_i \ln (K_r)^i_i - \Gamma_i = -\Gamma_i.$$

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Recently was proved (Błaszak et. al 2013) that for such "natural" quantization, Robertson and pre-Robertson conditions are satisfied only by Stäckel systems from Benenti class, for which

$$\Gamma_i = -\frac{1}{2} \frac{f'(\lambda_i)}{f(\lambda_i)} .$$

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Then, quantum Hamiltonians separates multiplicatively $\Psi = \prod_i \psi(\lambda_i)$ where $\psi(\lambda)$ solves the one-dimensional problem

$$(E_1 \lambda^{n-1} + E_2 \lambda^{n-2} + \cdots + E_n)\psi = -\frac{1}{2}\hbar^2 \left(f \frac{\mathrm{d}^2 \psi}{\mathrm{d}\lambda^2} + \frac{1}{2} f' \frac{\mathrm{d}\psi}{\mathrm{d}\lambda} \right) + \sigma(\lambda)\psi$$

Quantum separability: new results

How to find a separable quantization for other classes?

Quantum separability: new results

How to find a separable quantization for other classes?

For any other class

$$H_1\lambda^{\gamma_1} + H_2\lambda^{\gamma_2} + \cdots + H_n = \frac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda),$$

there exist a family of metric tensors $\bar{G}_\theta$ of the form

$$\bar{G}_\theta = \varrho_\gamma^{\frac{n}{2}}(\lambda_1, ..., \lambda_n) G_{B,\theta}$$

where $G_{B,\theta}$ is arbitrary metric tensor from Benenti class, for which a generalized Robertson condition is fulfilled and hence, again quantum Hamiltonians separates multiplicative with a common eigenfunctions $\Psi = \prod_i \psi(\lambda_i)$ where $\psi(\lambda)$ solves the one-dimensional problem

$$(E_1\lambda^{\gamma_1} + E_2\lambda^{\gamma_2} + \cdots + E_n)\psi = -\frac{1}{2}\hbar^2 f \left[\frac{\mathrm{d}^2\psi}{\mathrm{d}\lambda^2} + \left(\frac{f'}{f} - \frac{1}{2}\frac{\theta'}{\theta}\right)\frac{\mathrm{d}\psi}{\mathrm{d}\lambda}\right] + \sigma(\lambda)\psi$$

Separable minimal quantizations

In arbitrary local canonical coordinate chart (x, p)

$$\hat{\hat{H}}_r = -\frac{1}{2}\hbar^2\bar{\nabla}_i A^{ij}\bar{\nabla}_j + V_r(x), \quad r = 1, \ldots, n,$$

are commuting, Hermitean operators in a Hilbert space $L^2(Q, |\det \overline{g}|^{1/2}dx)$,where $\overline{g} = \rho^{\frac{2}{n}}(x)g_{B,\theta}(x)$.

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These operators can be represented in any other Hilbert space $L^2(Q, |\det g|^{1/2} dx)$ by

$$\hat{H}_r = U \hat{\bar{H}}_r U^{-1} = -\frac{1}{2}\hbar^2 \nabla_i A^{ij} \nabla_j + V_r(x) + \hbar^2 W_r(x), \quad r = 1, \ldots, n,$$

where isometry $U = |\det g|^{1/4} |\det \overline{g}|^{-1/4}$ and $\hbar^2 W_r(x)$ is some quantum deformation of the potential.
Hermitean operators $\hat{H}_r$ commute but do not separate in $L^2(Q, |\det g|^{1/2} dx)$.

Separable minimal quantizations

The explicit form of quantum correction

$$W(x) = \frac{1}{8}\left[A^{ij}\left(\overline{\Gamma}^k_{ik}\overline{\Gamma}^s_{js} - \Gamma^k_{ik}\Gamma^s_{js}\right) + 2\partial_i\left(A^{ij}\left(\overline{\Gamma}^k_{jk} - \Gamma^k_{jk}\right)\right)\right]$$

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In a special case, when $\overline{g} = \rho^{\frac{2}{n}}(x)g_{B,f}(x)$ and $g = g_{B,f}(x)$, the correction term attains the form

$$W(x) = \frac{1}{4}A^{ij}\left[\partial_i + \frac{1}{2}\left(\partial_i \ln \rho\right)\right]\left(\partial_j \ln \rho\right)$$

Example

Quantum separability of the system non-separable according to Robertson-Eisenhart model

Consider the Stäckel system for $n = 3$ given by the separation relations of the form:

$$H_1\lambda_i^3 + H_2\lambda_i + H_3 = \frac{1}{2}\lambda_i\mu_i^2 + \lambda_i^4, \quad i = 1, 2, 3$$

so that $\gamma_1 = 3$, $\gamma_2 = 1$ and $\gamma_3 = 0$ and with $f(\lambda_i) = \lambda_i$ and $\sigma(\lambda_i) = \lambda_i^4$. In this case $\rho_\gamma(\lambda) = -(\lambda_1 + \lambda_2 + \lambda_3)$.

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Consider also the corresponding flat metric G_B. In the coordinates x_1, x_2, x_3 defined through

$$\begin{aligned}(\lambda_1 + \lambda_2 + \lambda_3) &= -x_1 \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 &= x_2 + \frac{1}{4}x_1^2 \\ \lambda_1\lambda_2\lambda_3 &= \frac{1}{4}x_3^2\end{aligned} \quad (3)$$

Example

G_B reads

$$G_B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so $\rho = x_1$ in x_i-coordinates and x_i are flat non-orthogonal coordinates for G_B.

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In flat coordinates we receive $H_r = A_r^{ij} y_i y_j + V_r(x)$ where y_i are momenta conjugate to x_i and where the tensors A_r have the form

$$A_1 = \begin{pmatrix} 0 & -\frac{1}{x_1} & 0 \\ -\frac{1}{x_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{x_1} \end{pmatrix}$$

Example

$$A_2 = \begin{pmatrix} 1 & \frac{1}{4}x_1 - \frac{x_2}{x_1} & 0 \\ \frac{1}{4}x_1 - \frac{x_2}{x_1} & -x_2 & -\frac{1}{2}x_3 \\ 0 & -\frac{1}{2}x_3 & \frac{3}{4}x_1 - \frac{x_2}{x_1} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & \frac{1}{4}\frac{x_3^2}{x_1} & -\frac{1}{2}x_3 \\ \frac{1}{4}\frac{x_3^2}{x_1} & \frac{1}{4}x_3^2 & -\frac{1}{4}x_1x_3 \\ -\frac{1}{2}x_3 & -\frac{1}{4}x_1x_3 & \frac{1}{4}x_1^2 + x_2 + \frac{1}{4}\frac{x_3^2}{x_1} \end{pmatrix}$$

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with the corresponding potentials

$$V_1(x) = -\frac{3}{4}x_1 + \frac{x_2}{x_1}$$

$$V_2(x) = \frac{1}{16}x_1^3 + \frac{1}{2}x_1x_2 + \frac{1}{4}x_3^2 + \frac{x_2^2}{x_1}$$

$$V_3(x) = -\frac{1}{16}x_1x_3^2 - \frac{1}{4}\frac{x_2x_3^2}{x_1}$$

Example

From our theory it follows that we can perform a separable quantization of this system in the conformally flat metric $G = \frac{1}{u} G_B$ with $u = \rho^{2/n} = x_1^{2/3}$. We obtain three commuting operators

$$\widehat{H}_r = -\frac{1}{2}\hbar^2 \nabla_i A_r^{ij} \nabla_j + V_r(x)$$

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In the separation coordinates (λ, μ) the separation relations for $\widehat{H}_r$ are represented by three copies of one-dimensional problem

$$(E_1\lambda^3 + E_2\lambda + E_3)\psi(\lambda) = -\frac{1}{2}\hbar^2 \left(\lambda \frac{d^2\psi(\lambda)}{d\lambda^2} + \frac{1}{2}\frac{d\psi(\lambda)}{d\lambda}\right) + \lambda^4\psi(\lambda)$$

Example

Let us now rewrite our operators in the Hilbert space $L^2(Q, dx)$ with the flat metric $\overline{G} = G_B$. From our theory it follows that a suitable way of doing it is to quantize our Hamiltonians H_r directly in the metric $\overline{G}$ after amending them by the quantum correction terms given by

$$W_1 = 0, \; W_2 = -\frac{3}{8}\frac{1}{x_1^2}, \; W_3 = -\frac{1}{8}\frac{1}{x_1}$$

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One can check by direct calculations that the operators

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do indeed commute, thus constitute a quantum integrable system. These operators are however not qunatum separable in $L^2(Q, dx)$.

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Hietarinta problem 1984

Thank you for the attention