Cluster Structures on Poisson-Lie Groups

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Cluster algebras, invented by S. Fomin and A. Zelevinsky, help to construct a rich family of such coordinate systems.

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- Pentagram map and generalizations (Ovsienko, Schwartz, Tabachnikov; Glick; G.-S.-T.-V., Khesin, Soloviev; Glick, Pylyavskii, Mari-Beffa,...)

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Property **4** requires *ad hoc* work

Cluster Transformation

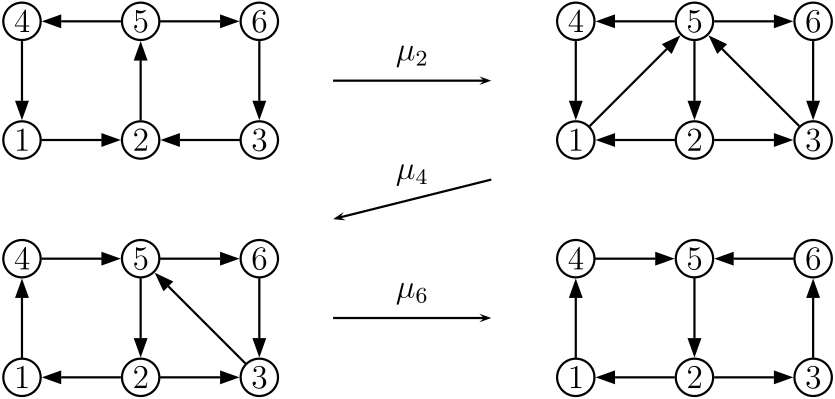
The *adjacent cluster* in direction $k \in [1, n]$:

$$x' = (\mathbf{x} \setminus \{x_k\}) \cup \{x'_k\},$$

where the new cluster variable x'_k is given by the *exchange relation*

$$x_k x'_k = \prod_{(i \to k) \in Q} x_i + \prod_{(k \to j) \in Q} x_j$$

Quiver Mutation



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- Grassmannians (J. Scott; GSV)
- Teichmüller spaces(GSV; Fomin-Shapiro-Thurston; Fock-Goncharov)
- ...

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- $\mathcal{G}$ is a simple complex Lie group;
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$$\{X \overset{\otimes}{,} X\}_r := [r, X \otimes X]$$

Example

For any matrix X we write its decomposition into a sum of lower triangular and strictly upper triangular matrices as

$$X = X_- + X_0 + X_+$$

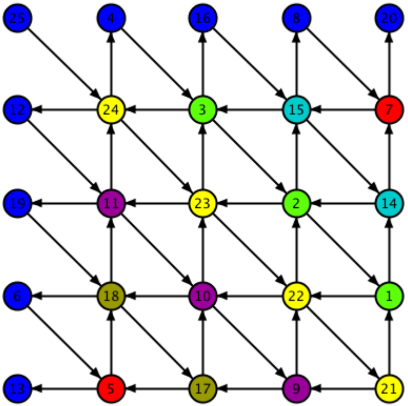
The standard R-matrix $R : Mat_n \to Mat_n$ defined by

$$R(X) = X_+ - X_-$$

The standard R-matrix Poisson-Lie bracket:

$$\{x_{ij}, x_{\alpha\beta}\}(X) = \frac{1}{2}(sign(\alpha - i) + sign(\beta - j))x_{i\beta}x_{\alpha j}$$

Standard cluster structure in GL_n: initial seed



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Belavin-Drinfeld Classification

Up to an automorphism, every *quasi triangular* classical R-matrix r belongs to one of disjoint classes $\mathcal{R}_T$ specified by the Belavin-Drinfeld data

$$T = (\Gamma_1, \Gamma_2, \tau), \quad (\Gamma_{1,2} \subset \Delta, \ \tau : \Gamma_1 \to \Gamma_2),$$

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where Δ is the set of simple positive roots and τ is an isometry s.t.

$$\forall \alpha \in \Gamma_1\ \exists m \in \mathbb{N}\ :\ \tau^j(\alpha) \in \Gamma_1\ (j = 0, \ldots, m-1),\ \tau^m(\alpha) \notin \Gamma_1\ .$$

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Rewrite $\{\cdot,\cdot\}_r$:

$$\{f_1, f_2\}_r = \langle R_\pm(\nabla^L f_1), \nabla^L f_2\rangle - \langle R_\pm(\nabla^R f_1), \nabla^R f_2\rangle,$$

where

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$$\langle\langle(\xi,\eta),(\xi',\eta')\rangle\rangle = \langle\xi,\xi'\rangle - \langle\eta,\eta'\rangle.$$

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Define subalgebras $\mathfrak{d}_\pm$ of $D(\mathfrak{g})$:

$$\mathfrak{d}_+ = \{(\xi,\xi)\colon \xi\in\mathfrak{g}\},\quad \mathfrak{d}_- = \{(R_+(\xi), R_-(\xi))\colon \xi\in\mathfrak{g}\}.$$

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- $(D(\mathfrak{g}), \mathfrak{d}_+, \mathfrak{d}_-)$ is *a Manin triple*
- $R_D = \pi_{\mathfrak{d}_+} - \pi_{\mathfrak{d}_-}$ defines a Poisson–Lie structure on $D(\mathcal{G}) = \mathcal{G} \times \mathcal{G}$ via

$$\{f_1, f_2\}_D = \frac{1}{2}\left(\langle\langle R_D(\nabla^L f_1), \nabla^L f_2\rangle\rangle - \langle\langle R_D(\nabla^R f_1), \nabla^R f_2\rangle\rangle\right),$$

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  where ∇^R and ∇^L are right and left gradients with respect to $\langle\langle\cdot,\cdot\rangle\rangle$.
- Restriction of this bracket to $\mathcal{G}$ identified with the diagonal subgroup of $D(\mathcal{G})$ (whose Lie algebra is $\mathfrak{d}_+$) coincides with the Poisson–Lie bracket $\{\cdot,\cdot\}_r$ on $\mathcal{G}$.

Main Conjecture

Let $\mathcal{G}$ be a simple complex Lie group.
For any Belavin-Drinfeld triple $T = (\Gamma_1, \Gamma_2, \tau)$ there exists a regular cluster structure $\mathcal{C}_T$ on $\mathcal{G}$ compatible with the corresponding Poisson-Lie bracket.

Example I: Standard Case

Trivial Belavin-Drinfeld data : $\Gamma_1 = \Gamma_2 = \varnothing$

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Standard Poisson-Lie Structure

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Cremmer-Gervais Poisson Structure

$$\mathcal{G} = SL_n$$
$$\Gamma_1 = \{\alpha_2, \dots, \alpha_{n-1}\},\ \Gamma_2 = \{\alpha_1, \dots, \alpha_{n-2}\}$$
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Theorem

There exists a cluster structure $\mathcal{C}_{CG}$ on $SL_n/GL_n/Mat_n$ compatible with the Cremmer–Gervais Poisson–Lie structure and satisfying all conditions of the Main Conjecture.

Table: Cremmer-Gervais vs. Standard Poisson-Lie bracket

	Standard	Cremmer-Gervais
$\{x_{11}, x_{55}\}$	$2x_{15}x_{51}$	$x_{15}x_{51} + x_{21}x_{45} + x_{25}x_{41} + x_{21}x_{45} + x_{31}x_{35}$
$\{x_{12}, x_{52}\}$	$x_{12}x_{52}$	$\frac{1}{5}x_{12}x_{52} + 2x_{22}x_{42} + x_{32}^2 - x_{11}x_{53} + x_{13}x_{51}$
$\{x_{15}, x_{51}\}$	$x_{12}x_{52}$	$-\frac{3}{5}x_{15}x_{51} + x_{21}x_{45} + x_{25}x_{41} + x_{31}x_{35}$

Initial Cluster

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For $X, Y \in Mat_n$, let

$$\mathcal{X} = \begin{bmatrix} X_{[2,n]} & 0 \end{bmatrix}, \quad \mathcal{Y} = \begin{bmatrix} 0 & Y_{[1,n-1]} \end{bmatrix}.$$

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Put $k = \lfloor \frac{n+1}{2} \rfloor$, $N = k(n-1)$ and define a $k(n-1) \times (k+1)(n+1)$ matrix

$$U(X,Y) = \begin{bmatrix} \mathcal{Y} & \mathcal{X} & 0 & \cdots & 0 \\ 0 & \mathcal{Y} & \mathcal{X} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{Y} & \mathcal{X} \end{bmatrix}.$$

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Define

$$\theta_i(X) = \det X^{[n-i+1,n]}_{[n-i+1,n]}, \ i \in [n-1];$$

$$\varphi_p(X,Y) = \det U(X,Y)^{[k(n+1)-p+1,k(n+1)]}_{[N-p+1,N]}, \ p \in [N];$$

$$\psi_q(X,Y) = \det U(X,Y)^{[k(n+1)-q+2,k(n+1)+1]}_{[N-q+1,N]}, \ q \in [M].$$

In the last family, $M = N$ resp. $M = N - n + 1$ if n is even resp. odd.

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The functions $\theta_i(X), \phi_p(X, Y), \psi_q(X, Y)$ *form a log-canonical family with respect to the Cremmer–Gervais bracket on the double of* GL_n*.*

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Intuition behind a construction of the initial cluster as well as the method of the proof come from expoiting certain invariance properties of functions on the double.

Initial Quiver

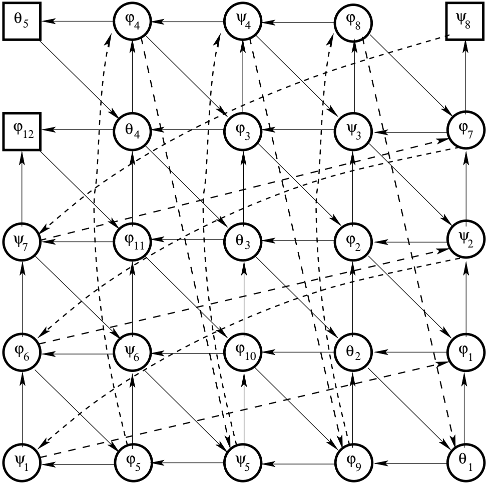

Figure: Quiver $Q_{CG}(5)$

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The proof relies on Desnanot-Jacoby-Dodgson-type identities applied to submatrices of $U(X, Y)$ while taking into account its shift-invariance properties.

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- $\mathcal{S}$ (# of mutations quadratic in n) - followed by freezing some of the cluster variables and localization at a single cluster variable $\varphi_{n-1}(X) = \det X^{[2,n]}_{[1,n-1]}$ - realizes a map

$$\zeta\colon \mathrm{Mat}_n \setminus \{X\colon \varphi_{n-1}(X) = 0\} \to \mathrm{Mat}_{n-1}$$

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- $\mathcal{T}$ (# of mutations cubic in n) - realizes the anti-Poisson involution $X \mapsto W_0 X W_0$ (W_0 - the longest permutation)

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Conjecture

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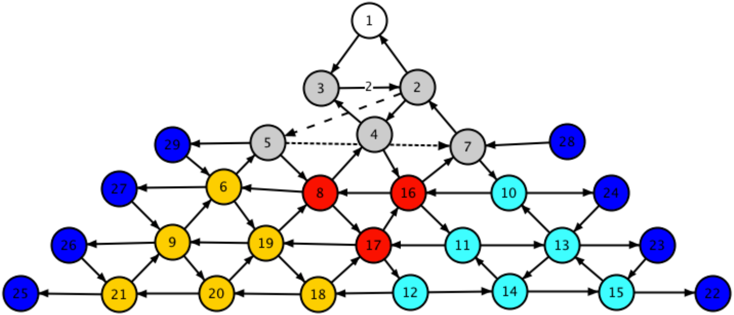
Proved for both the standard and Cremmer-Gervais cases in GL_n.

General GL_n case: proof in progress.

Initial quiver for the standard double of GL_n:



Initial cluster for the standard double of GL_n:

Left: $\det X_{[i,n]}^{[j,j+n-i]}$

Right: $\det Y_{[i,i+n-j]}^{[j,n]}$

Middle: $\det \begin{bmatrix} X^{[n-k+1,n]} & Y^{[n-l+1,n]} \end{bmatrix}_{[n-k-l+1,n]}$

Top:

$$\det(X)^* \times \det \begin{bmatrix} (\mathbf{1})^{[n-k+1,n]} & U^{[n-l+1,n]} & (U^2)^{[n]} & \dots & (U^{n-k-l+1})^{[n]} \end{bmatrix},$$

where $U = X^{-1}Y$.

Generalized exchange relation

Lemma

Let A *be an* $n \times n$ *matrix. For* $u, v \in \mathbb{C}^n$*, define matrices*

$$\Gamma(u) = \left[u\ Au\ A^2u \dots A^{n-1}u\right],$$

$$\Gamma_1(u,v) = \left[v\ u\ Au \dots A^{n-2}u\right],\ \ \Gamma_2(u,v) = \left[Av\ u\ Au \dots A^{n-2}u\right].$$

Let w *be the last row of the classical adjoint of* $\Gamma_1(u,v)$*, i.e.*

$$w\Gamma_1(u,v) = \left(\det \Gamma_1(u,v)\right) e_n^T\ .$$

Define $\Gamma^*(u,v)$ *to be the matrix with rows* $w, wA, \dots, wA^{n-1}$*. Then*

$$\det\left(\det \Gamma_1(u,v)A - \det \Gamma_2(u,v)\mathbf{1}\right) = (-1)^{\frac{n(n-1)}{2}} \det \Gamma(u) \det \Gamma^*(u,v).$$

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Thank you!