Integrability of the Szekeres System

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Abstract

The Szekeres system is a four-dimensional system of first-order, ordinary differential equations, with nonlinear, but polynomial (quadratic) right hand side. It is a reduction of the Einstein equations, related to a inhomogeneous spacetime with no symmetries.

The poster presents the way to solve it and to find its three independent global first integrals via Darboux polynomials and Jacobi's last multiplier method.

Derivation of the system

- Silent Universe [BMP]: dust spacetime, no rotation ($\omega_{ab} = 0$), Weyl tensor purely electric ($H_{ab}(\vec{u}) = 0$)
- Hydrodynamical formalism:
  - energy density ρ,
  - expansion rate Θ,
  - shear rate σ_{ab},
  - electric part of Weyl tensor E_{ab},
  - cosmological constant $\Lambda = 0$ or $\Lambda \neq 0$
- the Silent Universe (SU) system: six first-order ODEs

$$\begin{cases} \rho' = -\Theta\rho \\ \Theta' = -\frac{1}{3}\Theta^2 - 2\sigma_1^2 - 2\sigma_1\sigma_2 - 2\sigma_2^2 - \frac{1}{2}\rho \\ \sigma_1' = \frac{2}{3}\sigma_2(\sigma_1+\sigma_2) - \frac{1}{3}\sigma_1^2 - \frac{2}{3}\Theta\sigma_1 - E_1 \\ \sigma_2' = \frac{2}{3}\sigma_1(\sigma_1+\sigma_2) - \frac{1}{3}\sigma_1^2 - \frac{2}{3}\Theta\sigma_1 - E_2 \\ E_1' = E_1(\sigma_1 - \sigma_2) - E_2(\sigma_1 + 2\sigma_2) - \Theta E_1 - \frac{1}{2}\rho\sigma_1 \\ E_2' = E_2(\sigma_2 - \sigma_1) - E_1(\sigma_2 + 2\sigma_1) - \Theta E_2 - \frac{1}{2}\rho\sigma_2 \end{cases}$$

- The Szekeres system is a reduced SU with $\sigma_1 = \sigma_2 = \sigma$ and $E_1 = E_2 = E$.

The Szekeres System [Sz]

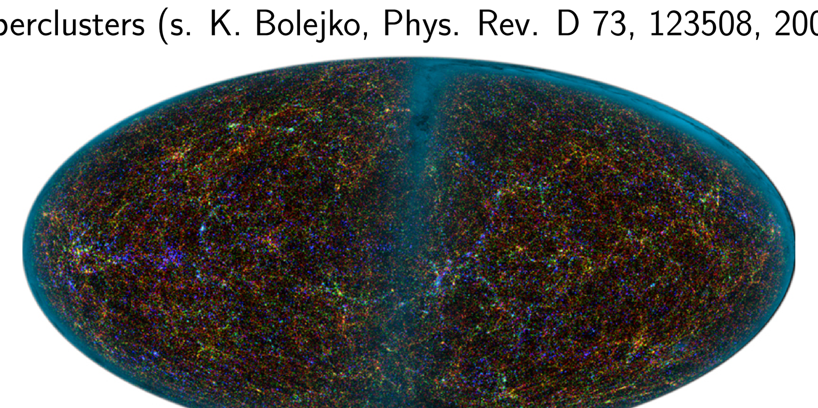
$$\begin{cases} \rho' = -\Theta\rho \\ \Theta' = -\frac{1}{3}\Theta^2 - 6\sigma^2 - \frac{1}{2}\rho\ (+\Lambda) \\ \sigma' = \sigma^2 - \frac{2}{3}\Theta\sigma - E \\ E' = 3E\sigma - \Theta E - \frac{1}{2}\rho\sigma \end{cases} \qquad \text{(Sz)}$$

Application

Dust models with inhomogeneous distribution of matter, such as:

- early Universe (s. A. P. Billyard, A. A. Coley, J. E. Lidsey, J. Math. Phys. 40, 5092, 1999)

- galaxy superclusters (s. K. Bolejko, Phys. Rev. D 73, 123508, 2006)



from: 2MASS galaxy catalog

Darboux polynomials method of finding first integrals of a polynomial system $x' = f(x)$

- Definition of a Darboux polynomial $J = J(x)$: there exists a polynomial $\mu = \mu(x)$ (a cofactor) such that $\delta_f J = \mu J.$
- Computation: fix $\deg J$, solve (overdetermined) linear equation for coefficients.
- For every Darboux polynomial J, a hyper-surface $J = 0$ is invariant under the flow δ_f.
- Sufficiently many (s. [G]) Darboux polynomials $J_1, \ldots J_n$ form a rational first integral of the form $I = J_1^{a_1} \cdot \ldots \cdot J_n^{a_n}$.

Two rational first integrals for (Sz)

- We found four independent Darboux polynomials for (Sz):

$$J_{11} = \rho,$$
$$J_{12} = 6E + \rho,$$
$$J_{21} = -18E + \Theta^2 - 3\rho + 6\Theta\sigma + 9\sigma^2,$$
$$J_{22} = 9E + \Theta^2 - 3\rho - 3\Theta\sigma - 18\sigma^2.$$

- They form two independent families of global first integrals, defined on common domain $\{J_{11} \cdot J_{12} \cdot J_{21} \cdot J_{22} \neq 0\}$. Let us fix two:

$$I_1 = \frac{-18E + \Theta^2 + 6\Theta\sigma - 3(\rho - 3\sigma^2)}{(6E+\rho)^{2/3}},$$
$$I_2 = \frac{(6E+\rho)^{1/3}(9E + \Theta^2 - 3\Theta\sigma - 3(\rho + 6\sigma^2))}{\rho}.$$

Change of variables $\kappa := 6E + \rho$, $\beta := 3\sigma + \Theta$

$$\begin{cases} \rho' = -\Theta\rho \\ \Theta' = -\Theta^2 - \frac{2}{3}\beta^2 + \frac{4}{3}\beta\Theta - \frac{1}{2}\rho \\ \beta' = -\frac{1}{3}\beta^2 - \frac{1}{2}\kappa \\ \kappa' = -\kappa\beta \end{cases} \qquad \text{(Sz}\kappa\beta\text{)}$$

First integrals in new variables $(\rho, \Theta, \beta, \kappa)$

$$i_1 = \frac{3\kappa - \beta^2}{\kappa^{2/3}}, \qquad i_2 = \frac{\sqrt[3]{\kappa}\,(4\beta^2 - 6\beta\Theta - 3\kappa + 9\rho)}{6\rho}.$$

Jacobi Multiplier method for a system $x' = f(x)$

- A Jacobi Multiplier (JM) is a non-zero C^1 function $M = M(x)$ so that

$$\mathrm{Div}(Mf) = 0.$$

- The system, multiplied by M, conserves its density.
- Jacobi's theorem [G]: if a n–dimensional system admits a Jacobi multiplier M and its $(n-2)$ independent integrals are known, then it admits an extra (last) first integral.

Jacobi Multiplier for Szekeres System

- We used Goriely's method for quasimonomial systems [G] to find the Jacobi multiplier for (Sz$\kappa\beta$):

$$M(\rho, \Theta, \beta, \kappa) = \frac{1}{\sqrt[3]{\kappa}\rho^3}.$$

- With M, i_1, i_2 the assumptions of the Jacobi's theorem are fulfilled, hence the last integral can be locally expressed by an integral.
- in new variables:

$$i_3 = -\frac{\sqrt{i_1^3}\sqrt[3]{\kappa}(\beta - 2\Theta)}{\rho} - \frac{3\beta\sqrt{i_1}\, i_2}{\kappa^{2/3}} - 3(3i_2 - i_1)\arctan\left(\frac{\beta}{\sqrt{i_1}\sqrt[3]{\kappa}}\right)$$

- Note that i_3 is defined on $\{i_1 > 0\}$, but one can extend it to a global first integral using arctanh and arccoth functions.

The last integral of (Sz)

for $J_{11} \cdot J_{12} \cdot J_{22} \neq 0, \quad J_{21} < 0$:

$$\frac{3\left(-E\left(18E + 2\Theta^2 + 3\rho\right) + 3\Theta\sigma(2E+\rho) + 9\sigma^2(4E+\rho)\right)\arctan\left(\frac{\Theta+3\sigma}{\sqrt{18E-(\Theta+3\sigma)^2+3\rho}}\right)}{\rho(6E+\rho)^{2/3}} + \\ + \frac{\sqrt{18E - (\Theta+3\sigma)^2 + 3\rho}\,\left(9E(\sigma - \Theta) + \sigma\left((\Theta+3\sigma)^2 + 6\rho\right)\right)}{\rho(6E+\rho)^{2/3}},$$

for $J_{11} \cdot J_{22} \neq 0, \quad J_{21} > 0, \quad J_{12} < 0$:

$$\frac{3\left(-E\left(18E + 2\Theta^2 + 3\rho\right) + 3\Theta\sigma(2E+\rho) + 9\sigma^2(4E+\rho)\right)\operatorname{arctanh}\left(\frac{\Theta+3\sigma}{\sqrt{-(18E-(\Theta+3\sigma)^2+3\rho)}}\right)}{\rho(6E+\rho)^{2/3}} - \\ - \frac{\sqrt{-(18E - (\Theta+3\sigma)^2 + 3\rho)}\,\left(9E(\sigma - \Theta) + \sigma\left((\Theta+3\sigma)^2 + 6\rho\right)\right)}{\rho(6E+\rho)^{2/3}},$$

for $J_{11} \cdot J_{22} \neq 0, \quad J_{21} > 0, \quad J_{12} > 0$:

$$\frac{3\left(-E\left(18E + 2\Theta^2 + 3\rho\right) + 3\Theta\sigma(2E+\rho) + 9\sigma^2(4E+\rho)\right)\operatorname{arccoth}\left(\frac{\Theta+3\sigma}{\sqrt{-(18E-(\Theta+3\sigma)^2+3\rho)}}\right)}{\rho(6E+\rho)^{2/3}} - \\ - \frac{\sqrt{-(18E - (\Theta+3\sigma)^2 + 3\rho)}\,\left(9E(\sigma - \Theta) + \sigma\left((\Theta+3\sigma)^2 + 6\rho\right)\right)}{\rho(6E+\rho)^{2/3}}.$$

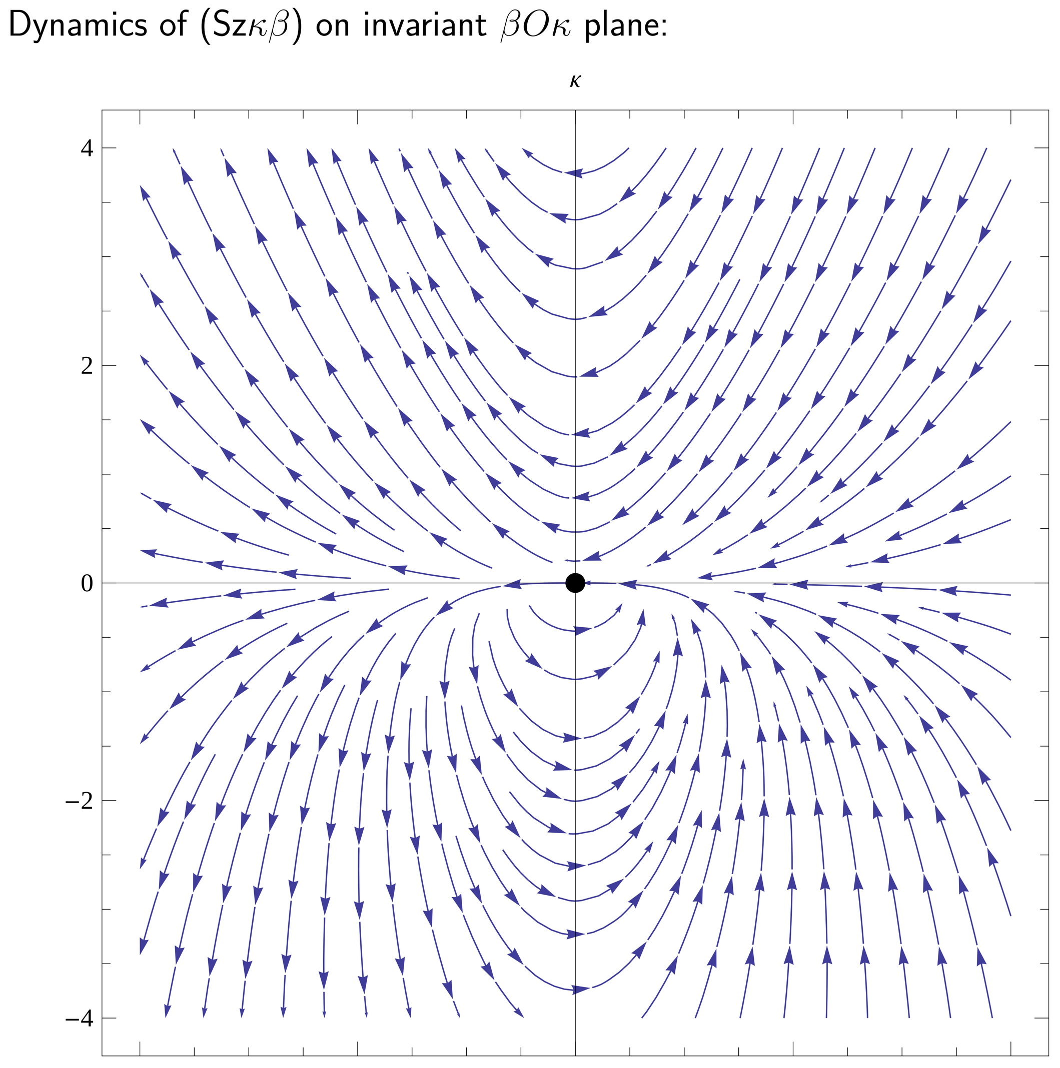
Direct solution of (Sz$\kappa\beta$)

- Two last equations of (Sz$\kappa\beta$) together with first integral i_1 depend on κ and β only.
- Setting $i_1 = \text{const}$ and $\beta = \pm\sqrt{3\kappa - i_1\kappa^{2/3}}$, we obtain an autonomous first order differential equation

$$\kappa'(t) = \mp\kappa\sqrt{3\kappa - i_1\kappa^{2/3}},$$

which can be solved in quadratures.
- This allows to find directly β and, using later i_2, also Θ and ρ.

Dynamics of (Sz$\kappa\beta$) on invariant $\beta O \kappa$ plane:



Further questions

- Is (Sz) integrable in the class of polynomials? (No)
- Is it integrable in the class of rational/analytic functions?
- Is Silent Universe system integrable? (Darboux polynomials form only one first integral, JLM method cannot be applied.)
- What about Szekeres system with cosmological constant $\Lambda \neq 0$? (see next column)

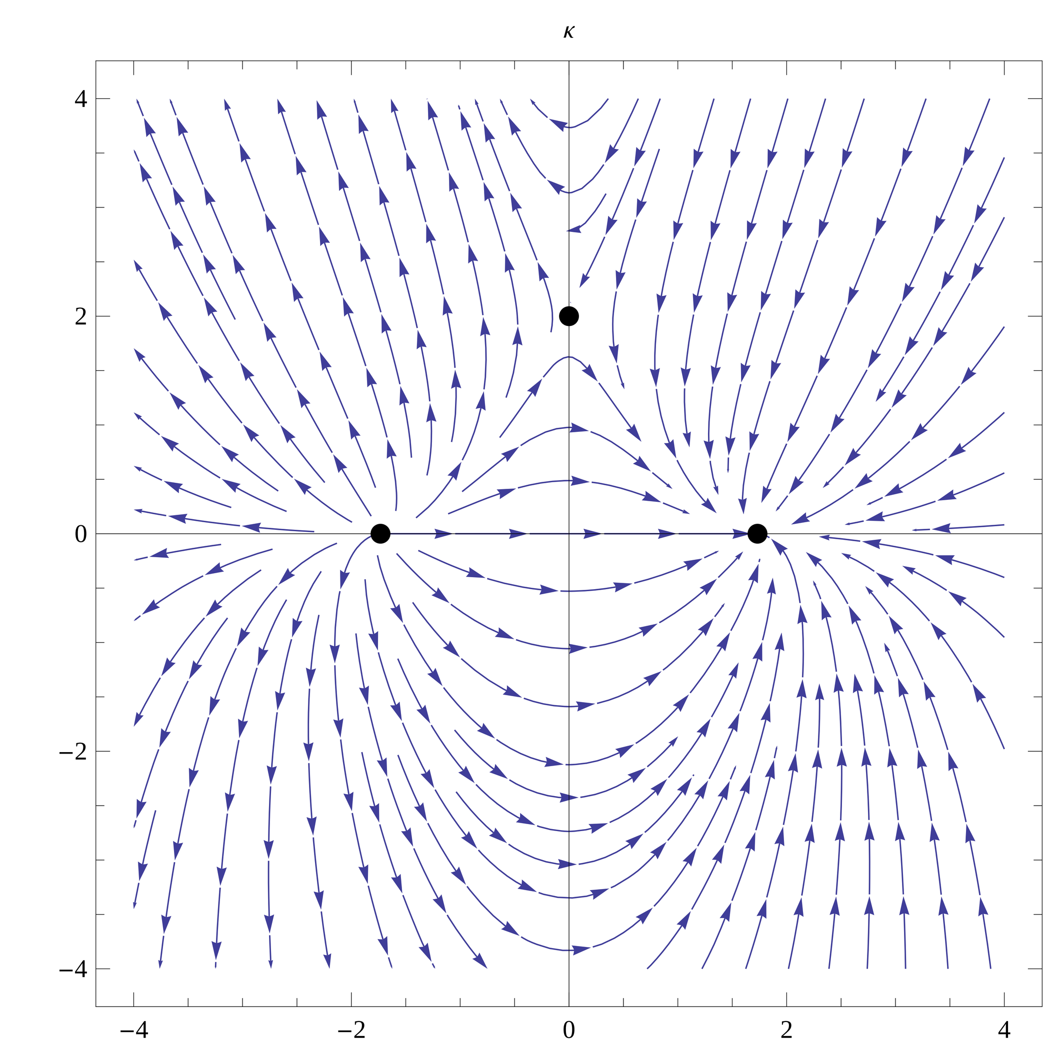
Szekeres system with cosmological constant [MB]

$$\begin{cases} \rho' = -\Theta\rho \\ \Theta' = \Lambda - \frac{1}{3}\Theta^2 - 6\sigma^2 - \frac{1}{2}\rho \\ \sigma' = \sigma^2 - \frac{2}{3}\Theta\sigma - E \\ E' = 3E\sigma - \Theta E - \frac{1}{2}\rho\sigma \end{cases} \qquad \text{(Sz}\Lambda\text{)}$$

Change of variables $\kappa := 6E + \rho$, $\beta := 3\sigma + \Theta$

$$\begin{cases} \rho' = -\Theta\rho \\ \Theta' = \Lambda - \Theta^2 - \frac{2}{3}\beta^2 + \frac{4}{3}\beta\Theta - \frac{1}{2}\rho \\ \beta' = \Lambda - \frac{1}{3}\beta^2 - \frac{1}{2}\kappa \\ \kappa' = -\beta\kappa \end{cases}$$

Dynamics of (SzΛ) with $\Lambda = 1$:



Integrability of (SzΛ)

- Similar Darboux polynomials – form similar two rational first integrals:

$$I_1 = \frac{-18E + \Theta^2 + 6\Theta\sigma - 3(\rho - 3\sigma^2) - 3\Lambda}{(6E+\rho)^{2/3}},$$
$$I_2 = \frac{(6E+\rho)^{1/3}(9E + \Theta^2 - 3\Theta\sigma - 3(\rho + 6\sigma^2) - 3\Lambda)}{\rho}.$$

- Similar change of variables (see above).
- $M = \dfrac{1}{\sqrt[3]{\kappa}\rho^3}$ is also a JLM in this case, so due to Jacobi's Theorem there exists the last integral (a huge formula; expressed by elliptic functions).

Conclusion

The Szekeres system is completely integrable for any $\Lambda \in \mathbb{R}$.

References

[BMP] M. Bruni, S. Matarrese, O. Pantano, *Dynamics of Silent Universes*, Astroph. J., 445: 958–977, 1995.

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