The trigonometric BC_n Sutherland system: action-angle duality and applications

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Abstract

Action-angle duality for the trigonometric BC_n Sutherland system is explored via Hamiltonian reduction [1]. Consequently, various features such as equilibrium, degeneracy, and connection to a family of commuting Hamiltonians found by van Diejen are elucidated [2].

Introduction

The trigonometric BC_n Sutherland system

The **trigonometric BC_n Sutherland system** is defined by the Hamiltonian

$$H(q,p)=\frac{1}{2}\langle p,p\rangle+\sum_{\alpha\in\mathrm{BC}_n^+}\frac{\gamma_\alpha}{\sin^2\langle\alpha,q\rangle},\qquad(1)$$

where $\langle .,.\rangle$ denotes the standard inner product on $\mathbb{R}^n$, α runs over the positive roots of root system BC_n, and γ_α are coupling constants depending only on the length of α. Hence there are **three independent parameters**, denoted by γ,γ_1,γ_2, and in order to ensure pure repulsion, are restricted as follows

$$\gamma>0,\quad\gamma_2>0,\quad 4\gamma_1+\gamma_2>0.\qquad(2)$$

The phase space is the cotangent bundle $T^*C_1=C_1\times\mathbb{R}^n$ of the ***Weyl alcove***

$$C_1=\big\{q\in\mathbb{R}^n\mid\pi/2>q_1>\cdots>q_n>0\big\},\qquad(3)$$

and q,p are Darboux coordinates, i.e. the canonical symplectic form on T^*C_1 is of the form

$$\omega=\sum_{j=1}^n dq_j\wedge dp_j.\qquad(4)$$

A physical interpretation of the trigonometric BC_n Sutherland model is depicted below.

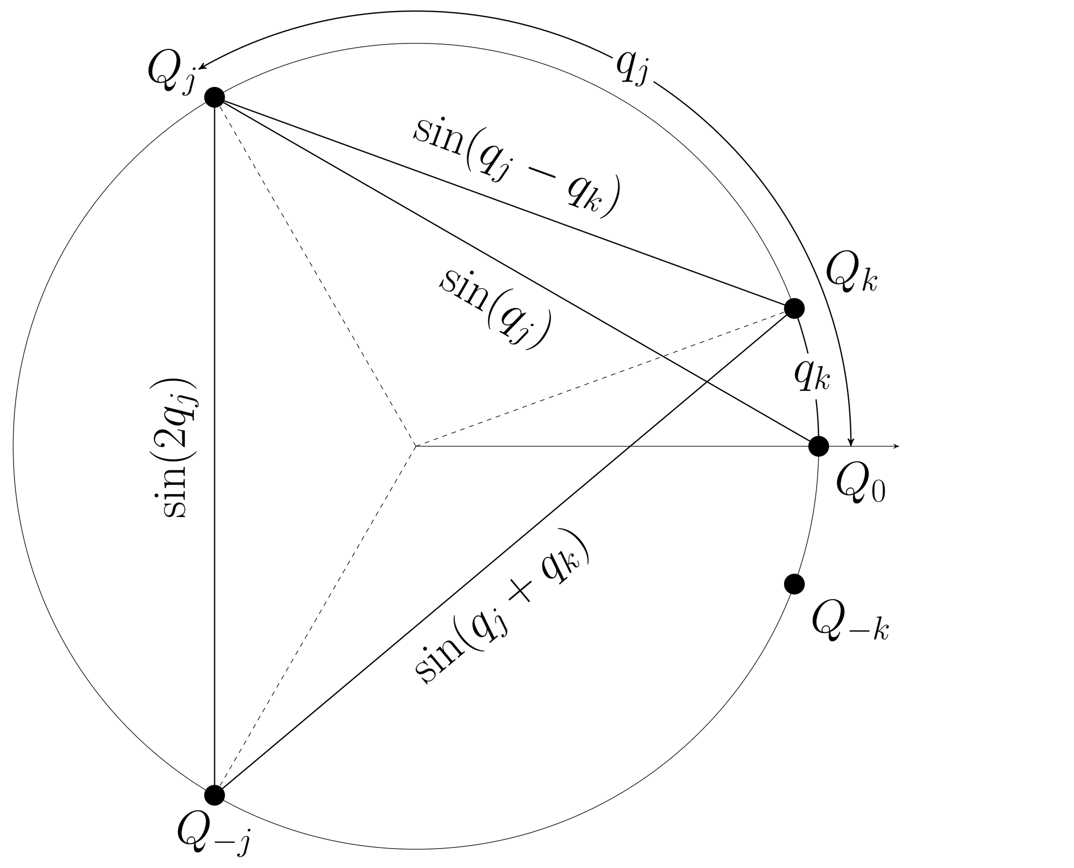

Figure 1: $2n+1$ particles move symmetrically w.r.t. a fixed point Q_0 on a circle of radius $r=1/2$. Interaction is given by a pair potential inversely proportional to the square of the chord-distance.

The dual system

At a 'semi-global' level, the dual system has the Hamiltonian

$$\tilde H^0(\lambda,\vartheta)=\sum_{j=1}^n\cos(\vartheta_j)|w(\lambda_j)|\prod_{\substack{k=1\\(k\neq j)}}^n|v(\lambda_j-\lambda_k)||v(\lambda_j+\lambda_k)|-\frac{\nu\kappa}{4\mu^2}\prod_{j=1}^n|v(\lambda_j)|^2+\frac{\nu\kappa}{4\mu^2},\qquad(5)$$

with potentials $v(z)=1+2\mathrm{i}\mu/z$, $w(z)=(1+\mathrm{i}\nu/z)(1+\mathrm{i}\kappa/z)$, and **coupling constants** μ,ν,κ satisfying

$$\mu>0,\qquad\nu>|\kappa|\geq0.\qquad(6)$$

Duality is established under the following relation of couplings

$$\gamma=\mu^2,\qquad\gamma_1=\frac{\nu\kappa}{2},\qquad\gamma_2=\frac{(\nu-\kappa)^2}{2}.\qquad(7)$$

The coordinates λ vary in a ***thick-walled Weyl chamber***

$$C_2=\big\{\lambda\in\mathbb{R}^n\mid\lambda_a-\lambda_{a+1}>2\mu\text{ and }\lambda_n>\nu\big\},\qquad(8)$$

and ϑ are angular variables. The Hamiltonian $\widetilde H^0$ generates dynamics via the symplectic form

$$\tilde\omega^0=\sum_{k=1}^n d\lambda_k\wedge d\vartheta_k.\qquad(9)$$

This system is a particular real form of the complex **rational BC_n Ruijsenaars – Schneider – van Diejen system.**

Action-angle duality

Duality via reduction – The basic idea

- Start with "big phase space", of group-theoretic origin, equipped with two canonical families of commuting "free" Hamiltonians.
- Apply suitable single (symplectic) reduction to the big phase space and construct two "natural" models, S and $\tilde S$, of the reduced phase space.
- The two families of "free" Hamiltonians become interesting many-body Hamiltonians and particle-positions in terms of both models. Their role interchanges in the two models.
- The natural symplectomorphism between the two models of the reduced phase space yields the action-angle map.

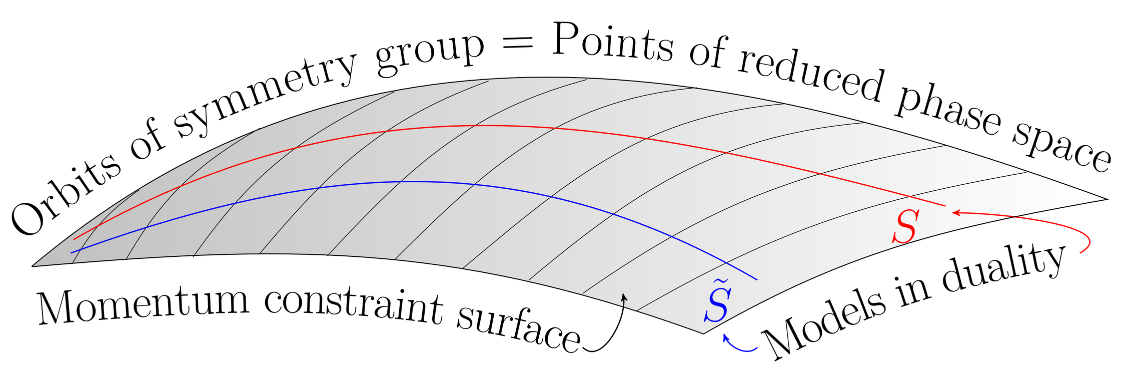

Figure 2: The geometry behind Hamiltonian reduction & action-angle duality.

Reduction on the unitary group $\mathrm{U}(2n)$

In [1] we started from the cotangent bundle of $\mathrm{U}(2n)$, i.e.

$$T^*\mathrm{U}(2n)\cong\mathrm{U}(2n)\times\mathfrak{u}(2n)=\{(y,Y)\},\qquad(10)$$

on which the fixed-point subgroup G_+ of the automorphism

$$y\mapsto CyC^{-1}\quad\text{with}\quad C=\begin{bmatrix}\mathbf{0}_n&\mathbf{1}_n\\\mathbf{1}_n&\mathbf{0}_n\end{bmatrix}\qquad(11)$$

acts smoothly, freely, and properly entailing that the quotient space of the constraint surface $J^{-1}(0)$ of the momentum map

$$J(y,Y,v^\ell,v^r)=\Big((yYy^{-1})_++v^\ell,-Y_++v^r\Big)\qquad(12)$$

is a smooth manifold. This is our reduced phase space

$$P_{\mathrm{red}}=J^{-1}(0)/(G_+\times G_+).\qquad(13)$$

Solving the momentum constraint

$$J(y,Y,v^\ell,v^r)=0\qquad(14)$$

by "diagonalizing"

1. the group component y, leads to a **global cross-section**

$$S=\{(e^{\mathrm{i}Q(q)},Y(q,p),v)\mid q\in C_1,\ p\in\mathbb{R}^n\}\qquad(15)$$

for the action of G_+ on $J^{-1}(0)$. Moreover, $Y(q,p)$ is a Lax matrix proving the trigonometric BC_n Sutherland system to be Liouville integrable. In particular, the spectral invariants

$$H_k(q,p)=\frac{(-1)^k}{4k}\mathrm{tr}(Y(q,p))^{2k},\quad k=1,\dots,n\qquad(16)$$

form a complete set of functions in involution, $H_1=H$ (1).

2. the Lie algebra part Y, gives another **cross-section**

$$\tilde S=\{(y(\lambda,\vartheta),\mathrm{i}(h\Lambda h^{-1})(\lambda),v)\mid\lambda\in C_2,\ e^{\mathrm{i}\vartheta}\in\mathbb{T}^n\}\qquad(17)$$

for the G_+-action restricted to a dense subset of $J^{-1}(0)$. A Lax matrix of the form $L(\lambda,\vartheta)=y(\lambda,\vartheta)^{-1}Cy(\lambda,\vartheta)C$ is obtained for the dual system. A Poisson commuting family is given by

$$\widetilde H_k(\lambda,\vartheta)=\frac{(-1)^k}{2k}\mathrm{tr}(L(\lambda,\vartheta))^k,\quad k=1,\dots,n\qquad(18)$$

with $\widetilde H_1=\widetilde H^0$ (5).

Remark. Introducing the complex variables

$$z_a=\sqrt{\lambda_a-\lambda_{a+1}-2\mu}\prod_{b=1}^a e^{\mathrm{i}\vartheta_b},\quad z_n=\sqrt{\lambda_n-\nu}\prod_{b=1}^n e^{\mathrm{i}\vartheta_b}\qquad(19)$$

enables one to complete the "semi-global" model $\tilde S$ of the dual system into a global model by allowing the zero value for the complex variables $z_1,\dots,z_n$. **This completion results from the symplectic reduction automatically.**

Applications

Equilibrium of the Sutherland system

The Sutherland Lax matrix is diagonalizable

$$Y(q,p)\sim\mathrm{i}\Lambda(\lambda)=\mathrm{i}\,\mathrm{diag}(\lambda,-\lambda),\qquad(20)$$

thus the action-angle transforms of (16) are of the form

$$h_k(\lambda)=\frac{\lambda_1^{2k}+\cdots+\lambda_n^{2k}}{2k},\quad k=1,\dots,n,\qquad(21)$$

and assume a **global minimum** in the closure of C_2

$$\min_{(q,p)\in C_1\times\mathbb{R}^n}H_k(q,p)=\min_{\lambda\in\overline{C_2}}h_k(\lambda)=h_k(\lambda^0),$$

at the boundary point $\lambda_a^0=(n-a)2\mu+\nu$, $a=1,\dots,n$. In terms of the "oscillator variables" $z\in\mathbb{C}^n$ the equilibrium $(q,p)=(q^e,0)$ of the Sutherland system corresponds to $z=0$.

Superintegrability of the dual system

The action-angle transforms of the Hamiltonians (18) are

$$\tilde h_k(q):=\frac{(-1)^k}{k}\sum_{j=1}^n\cos(2kq_j),\quad k=1,\dots,n.\qquad(22)$$

The dual model is **maximally superintegrable**, i.e. there are $(n-1)$ additional constants of motion of the form

$$f_i(q,p):=\sum_{j=1}^n p_j(X^{-1}(q))_{j,i},\quad i=2,\dots,n,\qquad(23)$$

where X is the $n\times n$ matrix

$$X_{a,b}=\frac{\partial\tilde h_a}{\partial q_b}=(-1)^{a+1}2\sin(2aq_b),\quad a,b=1,\dots,n.\qquad(24)$$

A simple calculation shows that

$$\det X(q)\propto\prod_j\sin 2q_j\prod_{j<k}(\cos 2q_k-\cos 2q_j).\qquad(25)$$

Hence $X(q)$ is invertible at every point $q\in C_1$.

Equivalence of two sets of Hamiltonians

A Poisson commuting family of functions F_ℓ $(\ell=1,\dots,n)$ involving the Hamiltonian (5) was found by van Diejen

$$F_\ell(\lambda,\vartheta)=\sum_{\substack{J\subset\{1,\dots,n\},\ |J|\leq\ell\\\varepsilon_j=\pm1,\ j\in J}}\cos(\vartheta_{\varepsilon J})V_{\varepsilon J;J^c}^{1/2}V_{-\varepsilon J;J^c}^{1/2}U_{J^c,\ell-|J|},\qquad(26)$$

with $\widetilde H^0=\frac{1}{2}F_1-n$, The coefficients K_m of the characteristic polynomial of the Lax matrix $L(\lambda,\vartheta)$, that is

$$\det(L(\lambda,\vartheta)-x\mathbf{1}_{2n})=\sum_{m=0}^{2n}K_m(\lambda,\vartheta)x^{2n-m},\qquad(27)$$

provide another complete set of integrals with $\widetilde H^0=-\frac{1}{2}K_1$.

Q: Are van Diejen's functions (26) and the spectral invariants in (27) related? (Non-trivial because $\widetilde H^0$ is superintegrable.)

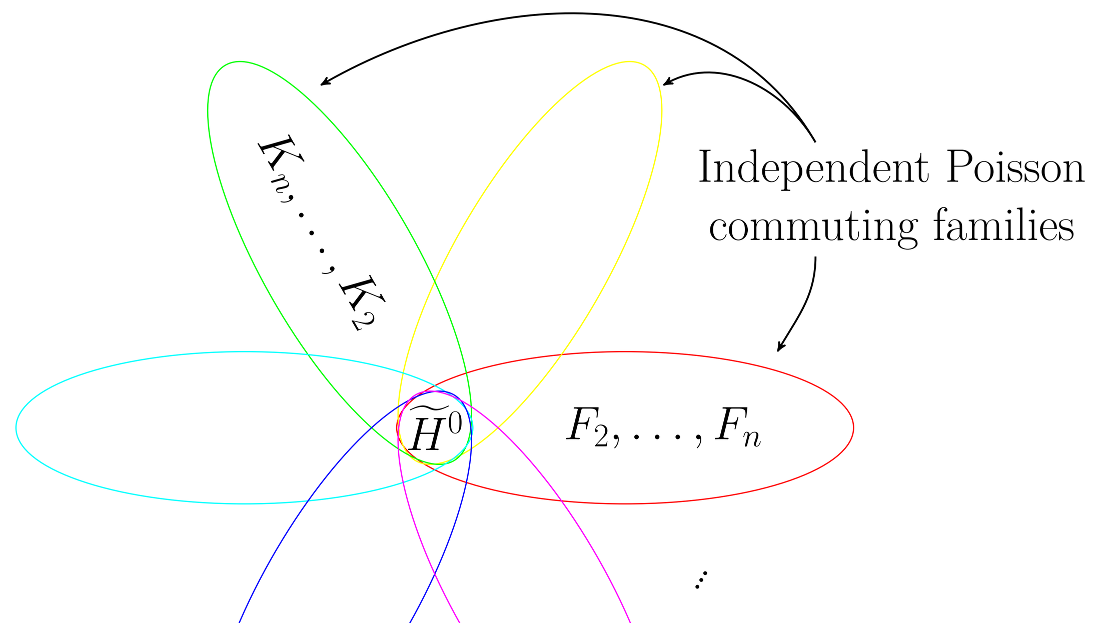

Figure 3: A possible, yet undesired scenario.

The affirmative answer is given in [2], where the following linear relation is proved

$$K_m(q)=(-1)^m\sum_{\ell=0}^m\binom{2(n-\ell)}{m-\ell}F_\ell(q).\qquad(28)$$

Our argument relies on the scattering theory of the rational BC_n RSvD system.

Results

In the framework of symplectic reduction we obtained a **Lax matrix** for the rational BC_n RSvD model with 3 independent parameters. **Action-angle duality** for the trigonometric BC_n Sutherland system with a global characterization of the phase spaces was constructed. The **equilibrium** of trigonometric BC_n Sutherland system was found. **Superintegrability** of the derived dual system and **equivalence** of the two families of Hamiltonians was proved.

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References

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