From semi-toric systems to Hamiltonian $\mathbb{S}^1$-actions and back

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Overview

1) Integrable systems: Definitions and classifications
   - Local normal form (Eliasson, Miranda & Zung)
   - Toric systems (Delzant)
   - Semi-toric systems (Pelayo & Vũ Ngọc)
   - Hamiltonian S^1-actions (Karshon)
2) From semi-toric systems to Hamiltonian S^1-actions and back:
   - General idea
   - Main results

Integrable Hamiltonian systems I

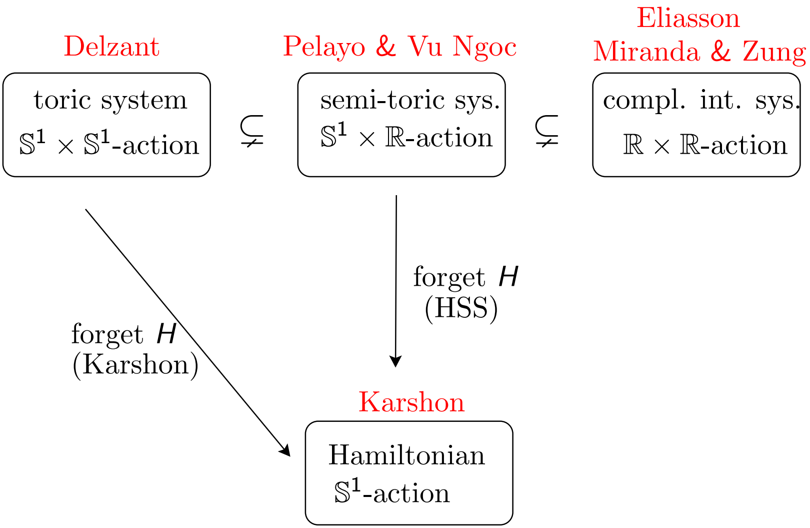
Setting

- (M, ω) 4-dim, compact, connected symplectic manifold
- A smooth $\Phi = (J, H) : M \to \mathbb{R}^2$ is a completely integrable Hamiltonian system if
  - Ham. vector fields X^J, X^H almost everywhere lin. independent
  - The Hamiltonian flows commute, i.e. $\varphi^J \circ \varphi^H = \varphi^H \circ \varphi^J$ $(\Longleftrightarrow \{J, H\} = 0)$

Notation:
Induced $\mathbb{R}^2$-action: $\mathbb{R}^2 \times M \to M, \quad (s, t).x := \varphi_s^J \circ \varphi_t^H(x)$

We assume all appearing actions to be faithful/effective.

Integrable Hamiltonian systems II



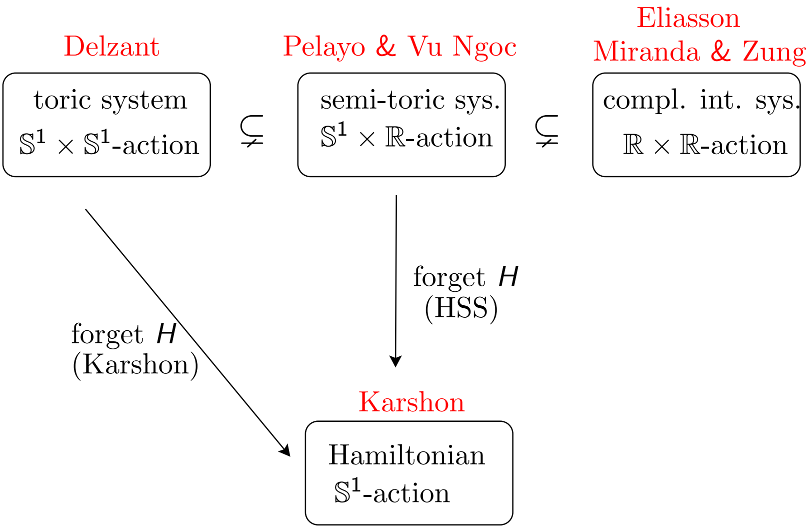
Integrable Hamiltonian systems III

Eliasson, Miranda & Zung (and others): Local normal form

Nice new coordinates (x, ξ) and new integrals q_j near nondegenerate critical point:

1) Hyperbolic component: $q_j(x, \xi) = x_j \xi_j$.
2) Elliptic component: $q_j(x, \xi) = \frac{1}{2}(x_j^2 + \xi_j^2)$.
3) Focus-Focus component (coming in pairs):
$$\begin{cases} q_{j-1}(x, \xi) = x_{j-1}\xi_j - x_j \xi_{j-1}, \\ q_j(x, \xi) = x_{j-1}\xi_{j-1} + x_j \xi_j. \end{cases}$$
4) regular component: $q_j(x, \xi) = \xi_j$.

Overview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$\subsetneq$
Pelayo & Vu Ngoc
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(Karshon)
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(HSS)
Karshon
Hamiltonian
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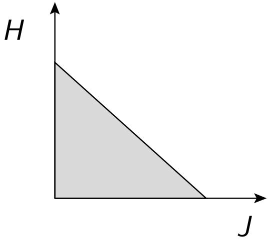
Toric systems

Delzant 1988: Toric classification

Let $\Phi = (J, H) : M \to \mathbb{R}^2$ be the moment map of an effective torus action. Then $\Phi(M) =: \Delta$ is a simple, convex, rational, smooth polygon, called Delzant polygon. And any such polygon determines a symplectic toric manifold (up to symplectomorphism preserving the moment map).

Example: $\Phi : \mathbb{CP}^2 \to \mathbb{R}^2$, $\quad \Phi(z_1 : z_2 : z_3) = \left(\frac{|z_1|^2}{\sum |z_i|^2}, \frac{|z_2|^2}{\sum |z_i|^2} \right)$



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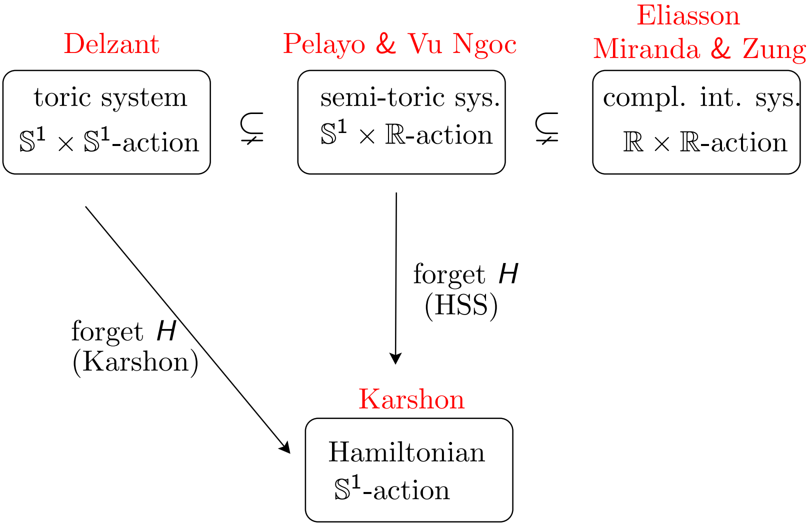
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Semi-toric systems I (Pelayo & Vũ Ngọc 2010)

Semitoric systems

A semitoric system $(M, \omega, \Phi = (J, H))$
is a completely integrable system with

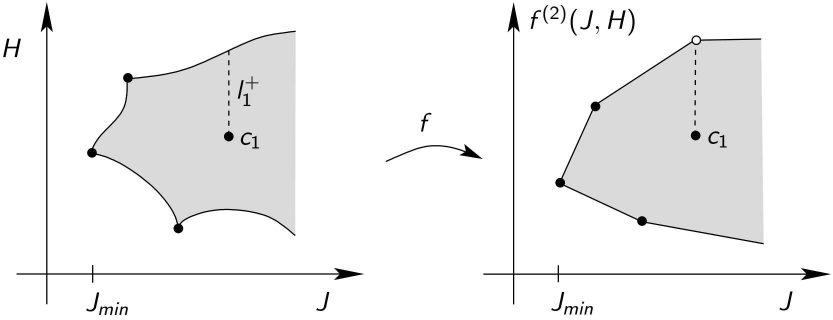
- $J : M \to \mathbb{R}$ induces an effective Hamiltonian S^1-action,
- only nondegenerate singularities,
- no hyperbolic singularities.

Conclusion: Possible singularities in dim $= 4$:

elliptic-elliptic or *elliptic-regular* or *focus-focus*

Semi-toric systems II (Pelayo & Vũ Ngọc 2010)

The image of a semi-toric Φ is a 'curved polygon' $\Phi(M)$ which can be 'straightened' to a polygon with cuts:

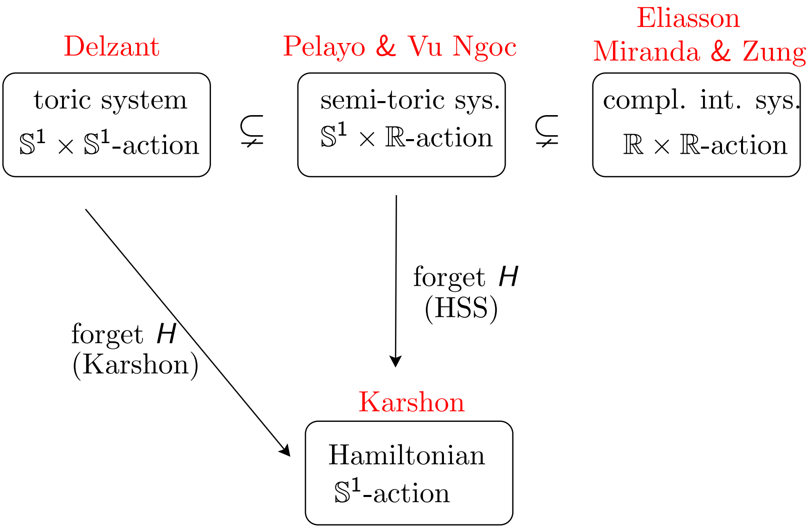


Semi-toric systems III (Pelayo & Vũ Ngọc 2010)

Classification invariants:

1. m_f, the number of focus-focus singularities.
2. An equivalence class of generalized polygons.
3. Singularity type invariant: Taylor series expansion of generating function at focus-focus points.
4. Volume invariant: Height of focus-focus value in polygon.
5. Twisting-index invariant: m_f numbers measuring ‘twistedness’ near focus-focus singularities.

Overview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Hamiltonian S^1-actions I

Karshon 1999:

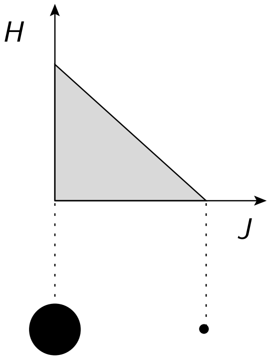
Let (M, ω) be a 4-dim sympl. manifold and let $J : M \to \mathbb{R}$ be the momentum map of an effective Hamiltonian S^1-action. Two such spaces are equivariantly symplectomorphic if and only if their associated labeled, directed graphs (see below) are equal.

Graph:

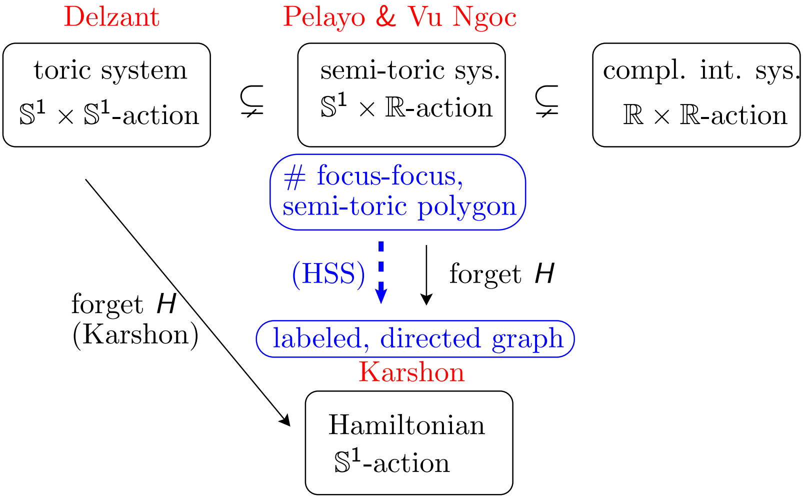
- Vertex set: Fixed point = vertex, Fixed surface = fat vertex
- Edge set: Directed edges between vertices stand for $\mathbb{Z}_k$-sphere, $k \geq 2$, which are connected components of $\{x \in M \mid Stab(x) = \mathbb{Z}/k\mathbb{Z}\}$.
- Labels: value of moment map; fixed surface: volume & genus

Hamiltonian S^1-actions II

Easy example: Delzant polygon of $\mathbb{CP}^2$ and 'forgetting H':



From semi-toric to Hamiltonian S^1-spaces: our work

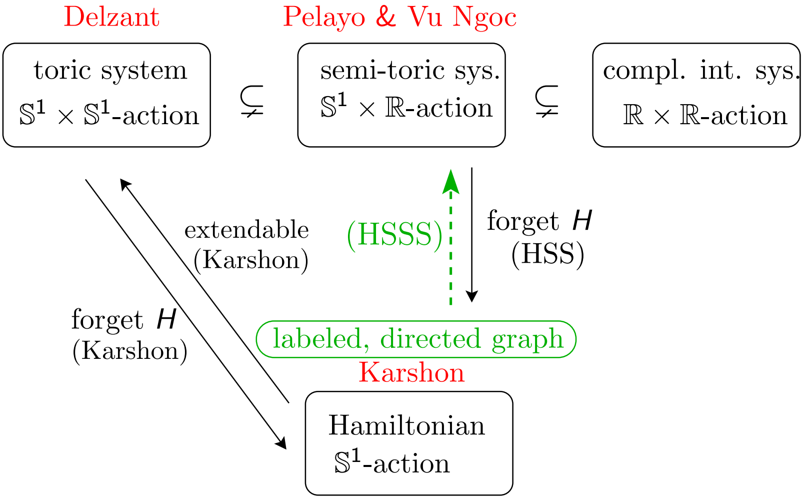


From semi-toric systems to Hamiltonian S^1-spaces:

Theorem (Hohloch & Sabatini & Sepe 2015)

1) Out of the **5** Pelayo & Vũ Ngọc-invariants, only **the first 2** are needed to recover the Karshon graph of the underlying Hamiltonian $\mathbb{S}^1$-space.
2) $(N, \omega, \Phi = (J, H))$ 'adaptable' $\Longleftrightarrow$ Its family of generalized semi-toric polygons contains a Delzant polygon.

From Hamiltonian S^1-actions to semi-toric systems:



From Hamiltonian S^1-actions to semitoric systems:

Theorem (Hohloch & Sabatini & Sepe & Symington)

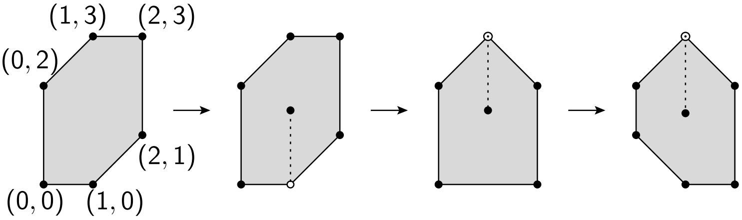
Let (M, J, ω) be a Hamiltonian $\mathbb{S}^1$-space with

1) fixed surfaces (if any) have genus zero,
2) each level set of J intersects at most two $\mathbb{Z}_k$-spheres.

Then there exists a smooth $H : M \to \mathbb{R}$ such that $(M, \omega, \Phi = (J, H))$ is an (essential) compact semitoric system.

Thank you for your attention!

Example: How to get a nonadaptable system



(a) (b) (c) (d)

The proof of the main theorem: Fixed surfaces and weights

Proposition 1 (HSS 2015)

Let $\Phi = (J, H)$ be semi-toric. Fixed surfaces of the S^1-action induced by J are symplectic spheres and they can only occur at the maximum or minimum of J.

Proposition 2 (HSS 2015)

Let $p \in M$ be an elliptic-elliptic fixed point of the semi-toric $\Phi = (J, H)$. Then the weights of the S^1-action at p can be computed from the 'straightened' polygon.

The proof of the main theorem: weights at focus-focus points I

Definition 3

$\Phi = (J, H)$ is called adaptable if the S^1-action of J extends to a Hamiltonian $S^1 \times S^1$-action.

Proposition 4 (HSS 2015)

Let $\Phi = (J, H)$ be adaptable. Then there are most two focus-focus points in each fiber of J and the weights of the induced S^1-action at the focus-focus points are $\{+1, -1\}$.

The proof of the main theorem: weights at focus-focus points and $\mathbb{Z}_k$-spheres

A local normal form argument allows to generalize Proposition 4 to

Proposition 5 (HSS 2015)

Weights at focus-focus points are always $\{+1, -1\}$.

Proposition 6 (HSS 2015)

The $\mathbb{Z}_k$-spheres correspond to the edges of the curved polygon $\Phi(M)$.

Smoothness of polygons

Recall: Definition 3

$\Phi = (J, H)$ is called adaptable if the S^1-action of J extends to a Hamiltonian $S^1 \times S^1$-action.

Definition 7

A vertex of a polygon is called smooth if its tangent vectors generate the lattice $\mathbb{Z}^2$.

Theorem (HSS 2015)

A semi-toric system admits a smooth 'straightened' polygon if and only if the system is adaptable.

Remark

Thus, adaptable semi-toric systems admit Delzant polygons!

Thank you for your attention!