# Topology of Integrable Hamiltonian systems on revolution surfaces

Elena Kantonistova,
Moscow State University

# Description of the problem

Consider a Riemmannian manifold $M \approx S^2$ with a metric $g$ which admits an $S^1$-action. We can write the metric $g$ in standard coordinates $(r, \varphi), r \in [0, L]$:

$$ds^2 = dr^2 + f^2(r)d\varphi^2,$$

where $f(r)$ is a smooth function, $f(0) = f(L) = 0$.

We call a pair $(M, g)$ a *manifold of revolution*.

Let us also consider a smooth *potential function* $V(r)$.

**Lemma.** *Consider the manifold of revolution* $(M, g)$*. If* $f(r) = -f(-r) = f(2L - r), f'(0) = 1, f'(L) = -1$ *and* $V(r) = V(-r) = V(2L - r)$ *then* $M$ *is smooth and* $f(r)$ *is a smooth function on it.*

Consider a natural mechanical system defined on $T^*M$ with standard symplectic structure $\omega$ on it and with a Hamiltonian function

$$H = \frac{1}{2} g^{ij}(x) p_i p_j + V(x),$$

where $(x^1, x^2)$ are the local coordinates on $M \approx S^2$, $(p_1, p_2)$ are impulses, i.e. the coordinates in $T_x^* M$, $(g^{ij}) = g^{-1}$.

**Definition.** *If $(M, g)$ is a manifold of revolution which satisfies conditions of Lemma 1, then the pair $\big(f(r), V(r)\big)$ defines a natural mechanical system on $(M, g)$.*

*Note that $(M, g)$ is not always embedded in $\mathbb{R}^3$.*

**Statement.** *The Hamiltonian system on a manifold of revolution is integrable for any pair $\big(f(r), V(r)\big)$.*

In coordinates $(r, \varphi)$ the first integrals have the form

$$H = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2f^2(r)} + V(r), \qquad K = p_\varphi.$$

**Definition.** The map $\phi\colon M \to \mathbb{R}^2\colon \phi(x) = (H(x), K(x))$ is called the *momentum map.* The image of the set of critical points $\{x \in M\colon rk\ d\phi(x) < 2\}$ under the momentum map is called the *bifurcation diagram*.

**Lemma 2.** *The natural mechanical system $(f(r), V(r))$, where $f'(r)^2 + V'(r)^2 > 0$, on the manifold of revolution $M$ has two critical points of rank $0$: $(0, N)$ and $(0, S)$ and a family of points of rank $1$ given by the formulas:*

$$h(r) = \frac{f(r)V'(r)}{f'(r)} + V(r), k(r) = \pm f(r)\sqrt{\frac{f(r)V'(r)}{f'(r)}}.$$

*The point $(0, N)$ (or $(0, S)$) is nondegenerate iff $V''(0) \neq 0$ and it has a center-center type if $V'' > 0$ and a focus-focus type if $sgn\, V'' < 0$.*

# Types of curves composing the bifurcation diagram

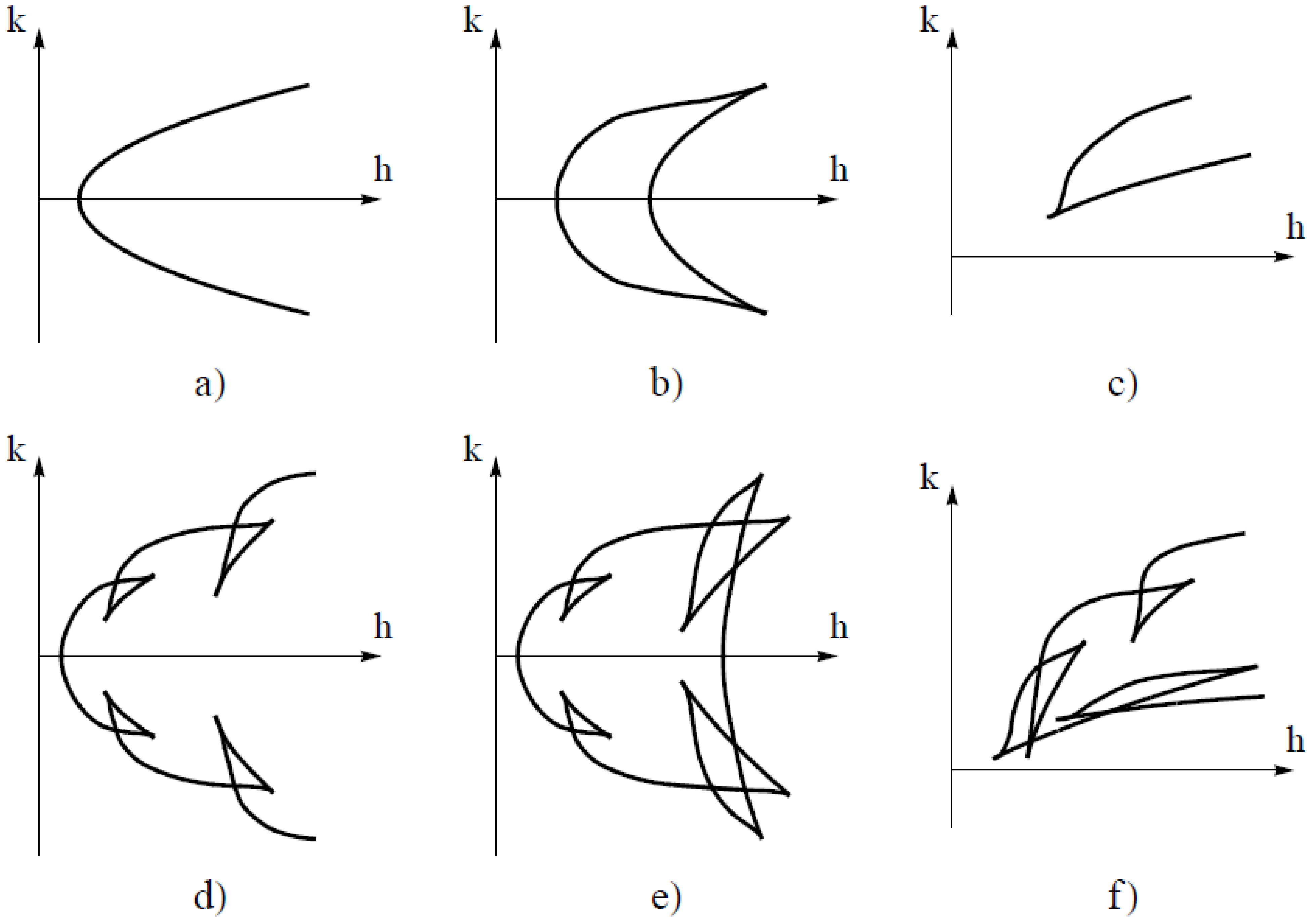
**Definition.** A curve $\gamma$ has a cusp singularity at a point $\mathrm{r}^*$ if

$$\gamma'(\mathrm{r}^*) = 0, \gamma''(\mathrm{r}^*) \neq 0, (\gamma'''(\mathrm{r}^*), \mathrm{w}) \neq 0,$$

where $|\mathrm{w}| = 1$, w is orthogonal to $\mathrm{v} := \frac{\gamma''(\mathrm{r}^*)}{|\gamma''(\mathrm{r}^*)|}$.

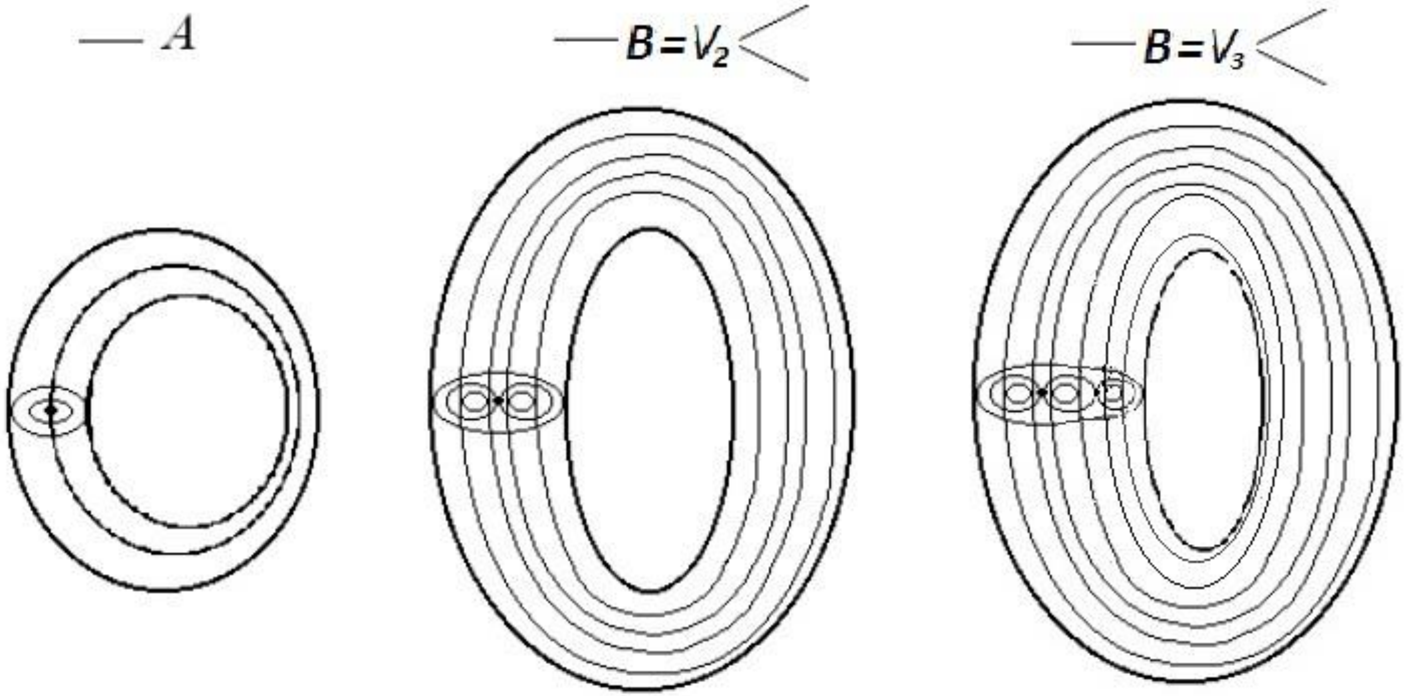
The bifurcation diagram of a system on a manifold of revolution consists of curves of three types – “parabolas”, “beaks”, and “moons”.

Each bifurcation curve can be simple (without cusp points) or singular (with cusp points).


k
h
a)
b)
c)
d)
e)
f)

# Atoms

- **Definition.** A Liouville equivalence class of a neighborhood of a singular fiber in a Liouville fibration is called a *3-atom*.
- In this work we need to know about atoms of type A, B, and $V_k$:

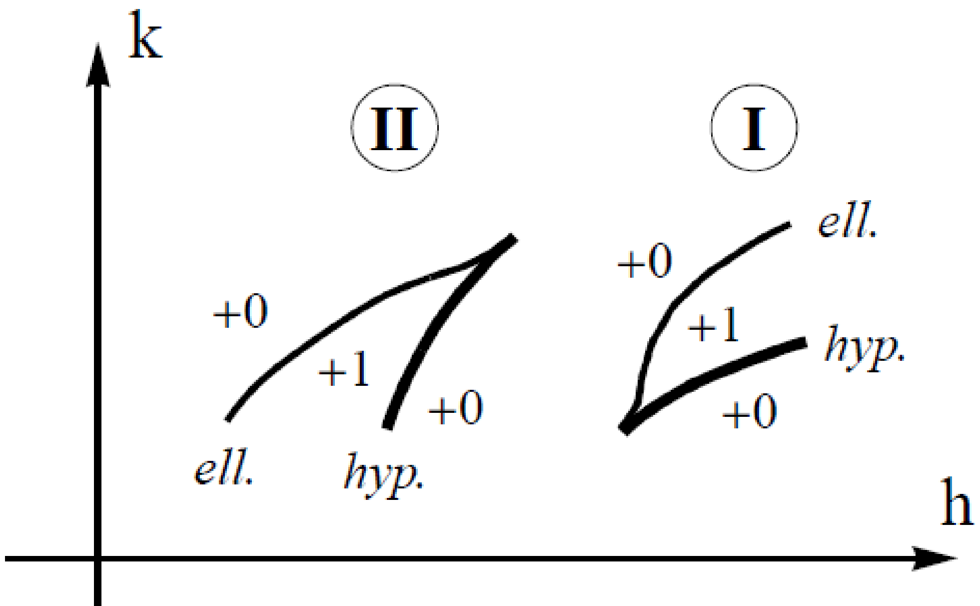


**Theorem 1.** *Consider a natural mechanical system* $\left(f(r), V(r)\right), r \in [0; L]$, *on the manifold of revolution* $M$. *If*

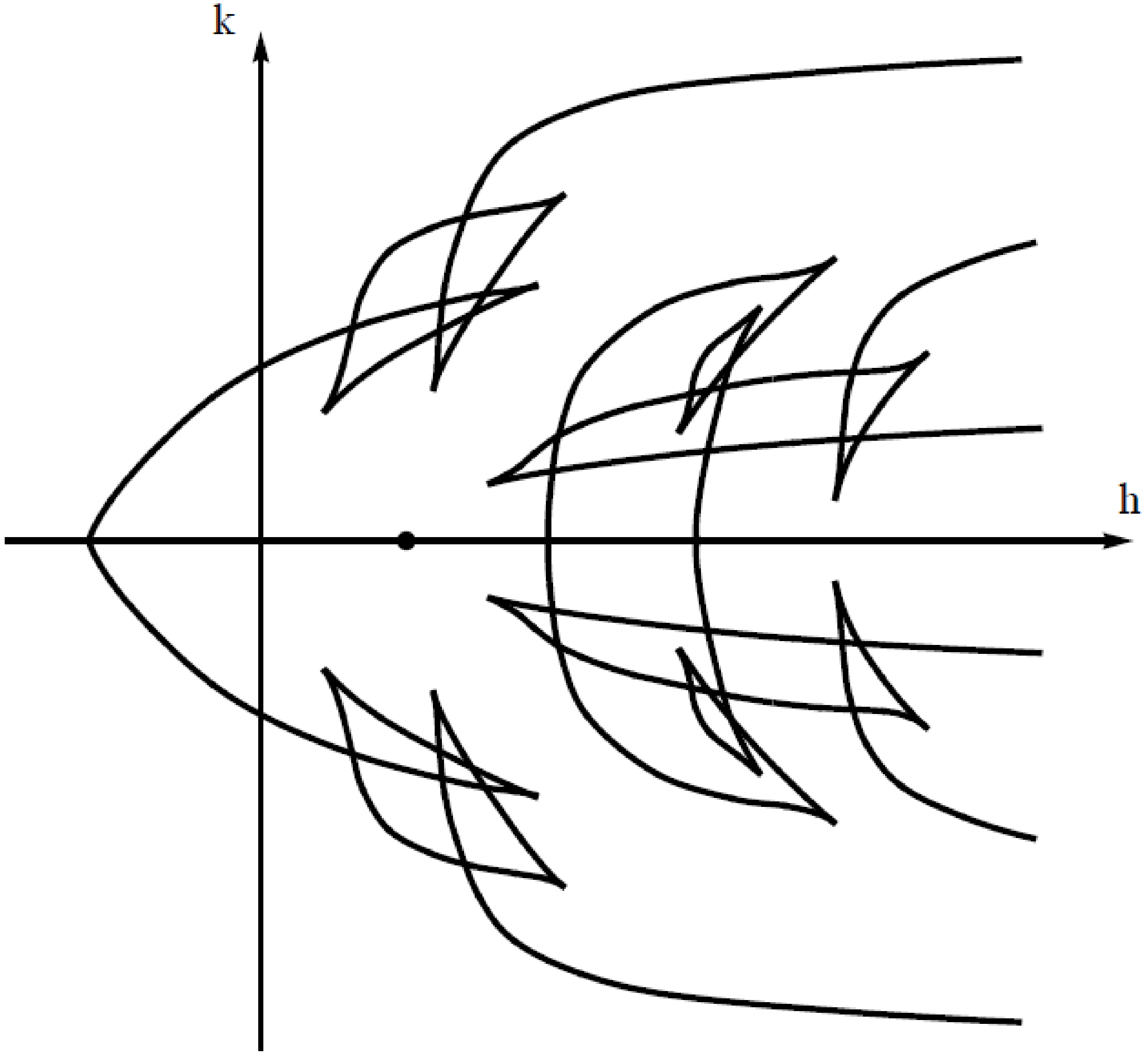
*a)* $f(r)$ *is a Morse function on* $(0; L)$ *and* $V(r)$ *is a Morse function at the endpoints of* $[0; L]$;

*b)* $f'(r)^2 + V'(r)^2 > 0, r \in (0; L)$, *then*

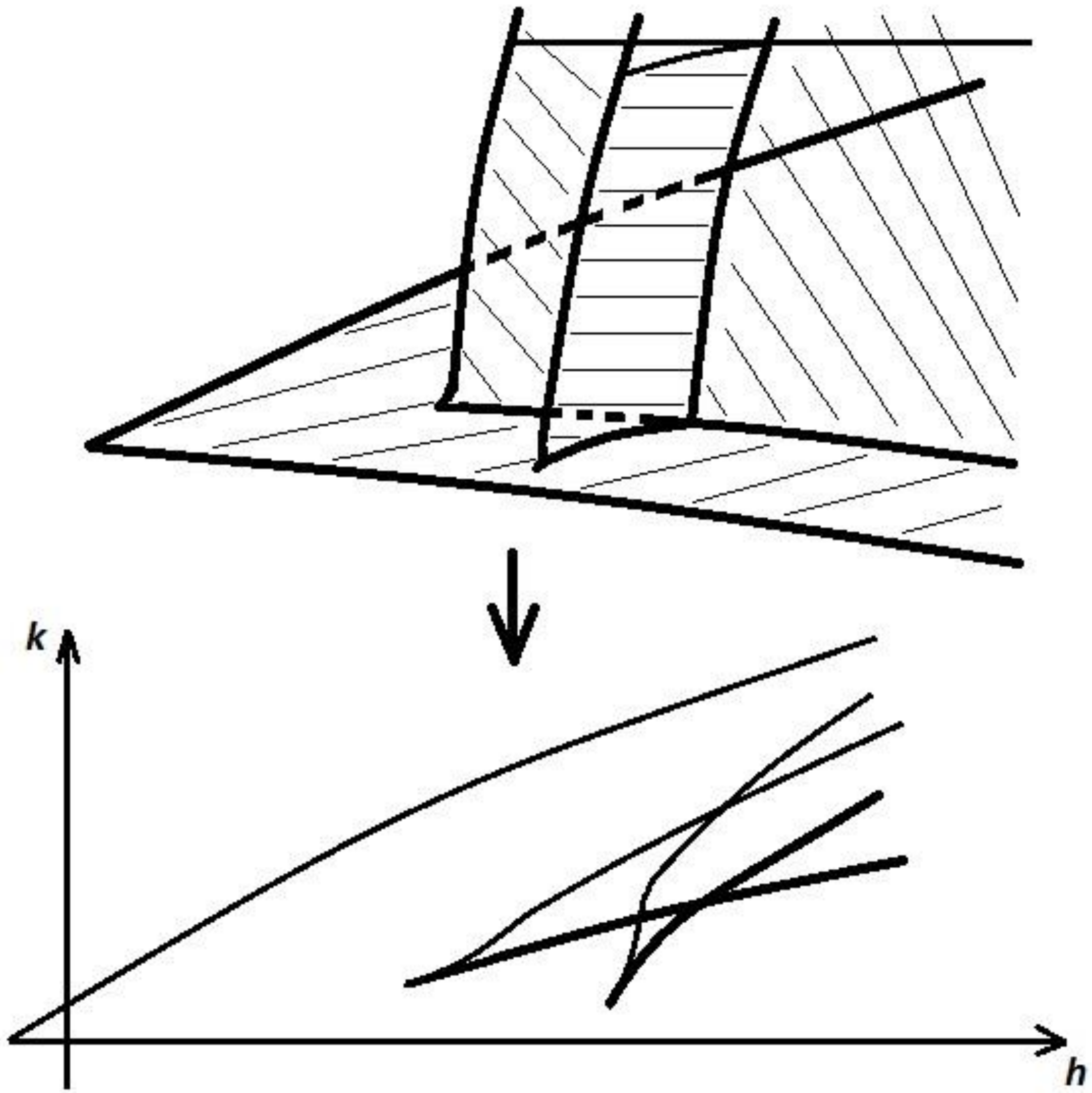
1) *every singular point of the bifurcation diagram is a cusp;*
2) *cusp points separate every bifurcation curve into regular subcurves of two types: A and B, which alternate along the curve. Near every cusp point the "left" subcurve has a B-type and the "right" subcurve has an A-type.*



# Example of a bifurcation diagram



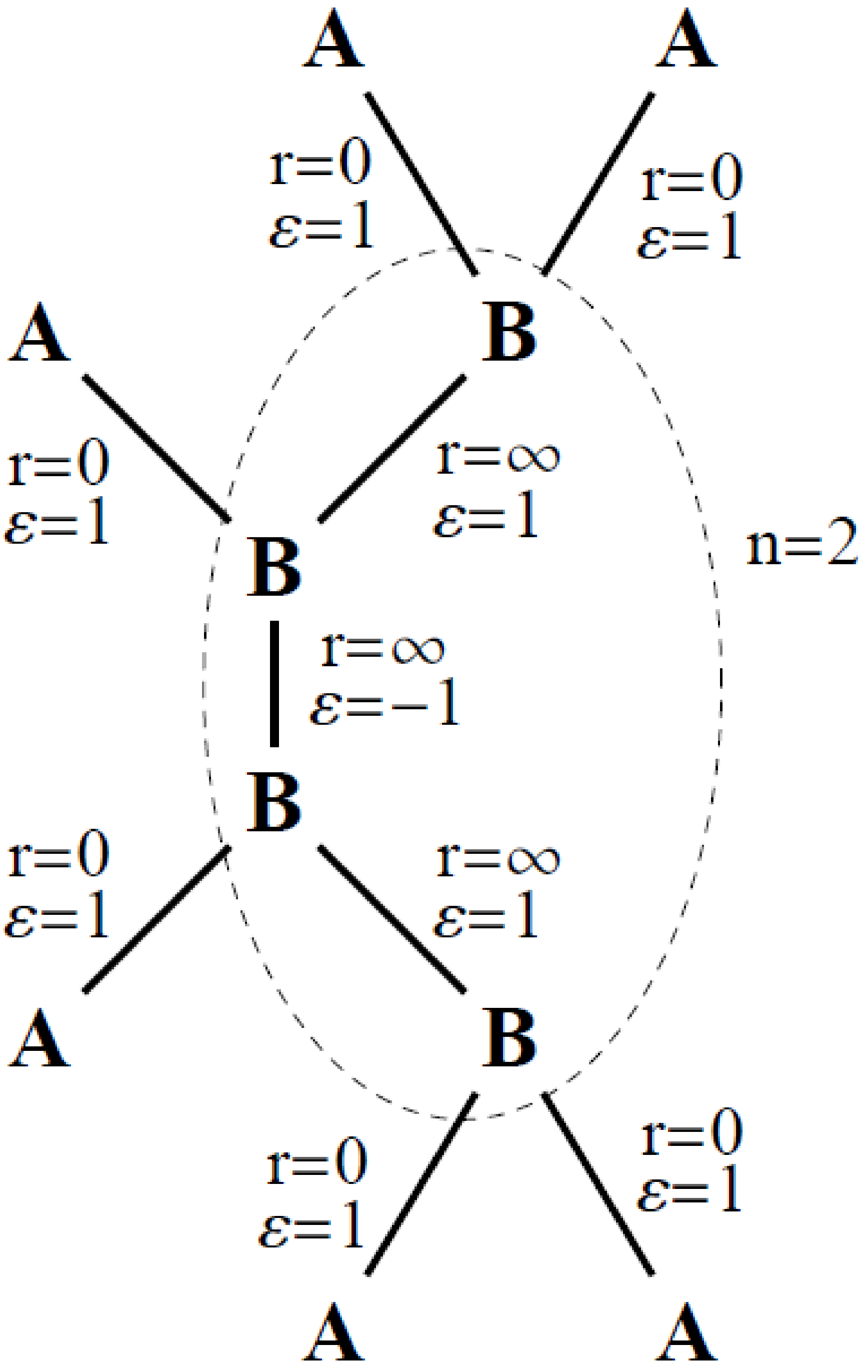
# The bifurcation complex



To every regular isoenergetic surface $Q_h^3$ corresponds a graph, that is the base of the Liouville fibration on $Q_h^3$. The interior points of this graph correspond to regular fibers of the Liouville fibration, while the vertices correspond to critical fibers (and corresponding atoms). We call this graph a *molecule*.

A molecule gives us important information about the structure of the Liouville fibration of $M$.

**Theorem 2.** *Consider a natural mechanical system* $(f(r), V(r))$ *on revolution manifold* $M$*, satisfying all conditions of Theorem 1. Let* $Q \subseteq Q_h^3$ *be a connected component of the isoenergetic surface* $Q_h^3$*. Then the molecule of the system on* $Q$ *is symmetric with respect to the h-axis*. *The molecule is of type* $A - A$ *or* $W - W$*, where* $W$ *has a tree structure with the root* $V_k$*, all interior vertices are also* $V_k$ *and all 1-valent vertices are of type* $A$*.*


A
A
r=0
ε=1
r=0
ε=1
B
A
r=0
ε=1
r=∞
ε=1
n=2
B
r=∞
ε=−1
B
r=0
ε=1
r=∞
ε=1
A
B
r=0
ε=1
r=0
ε=1
A
A

We can add some information to the molecule. On each edge of the molecule we can write certain numbers (called *marks)*. The molecule endowed with these marks is called a *marked molecule*.

**Theorem (Fomenko, Zieschang).** *Two non-degenerate integrable Hamiltonian systems are called Liouville equivalent if and only if their marked molecules coincide.*

**Theorem 3.** *Let $Q \subseteq Q_h^3$ be a connected component of an isoenergetic surface $Q_h^3$ of a system on a manifold of revolution. Let $W$ be the molecule of this system. Then*

*a) on the $edges\ A - V_k$ of $W$ the marks are: $r = 0, \varepsilon = +1$;*

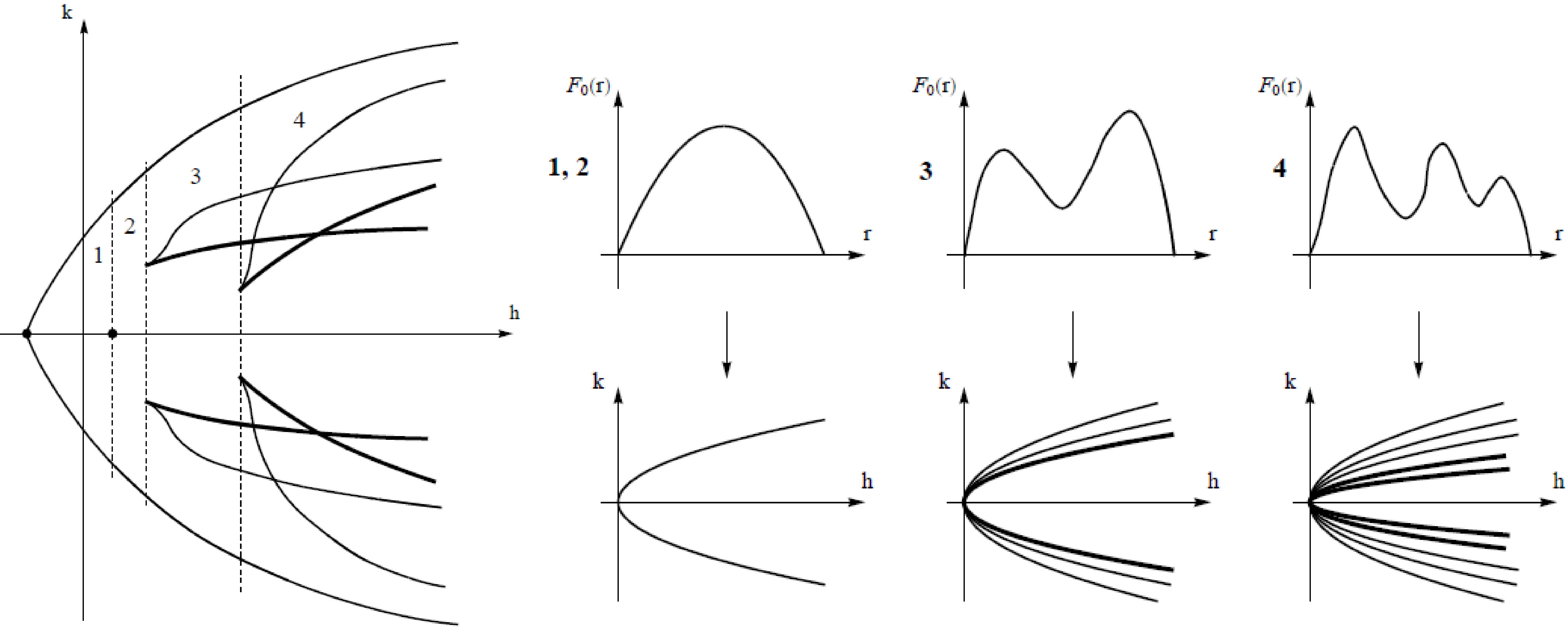
*b) on the $edges\ V_k - V_l$ of $W$ the marks are: $r = \infty, \varepsilon = +1\ (if\ the\ edge\ is\ symmetric\ with\ respect\ to\ H, then\ \varepsilon = -1)$;*

*c) on the $edges\ A - A$ of $W$ the marks are: $r = \frac{1}{2}, 0\ or\ \infty, \varepsilon = +1$;*

*d) the mark $n$ can be equal to $0, 1$ or $2$ (it is the number of critical points of rank $0$ on the corresponding fiber).*

# Liouville equivalence to geodesic flows

**Theorem 4.** *Consider the integrable system* $(f(r), V(r))$ *on a revolution manifold* $M$*. We can divide the set of values of* $h$ *into zones, whose boundaries are critical values of* $h$*. In every zone the system can be modelled by a geodesic flow (for an appropriate function* $f_0(r)$*). It means that the system* $(f(r), V(r))$ *is a "composition" of simple geodesic flows.*



k
h
1
2
3
4
1, 2
3
4
$F_0(r)$
r
k
h