ON INTRINSICAL CLOTHING OF THREE-REGULAR FAMILY OF HYPERPLANE ELEMENTS IN PROJECTIVE SPACE

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Abstract

In multidimensional projective space a family $\mathbb{B}$ of hyperplane elements is considered. The problem of constructing a composite clothing intrinsically attached to such a family is posed. The solution of this problem is obtained in the general case characterized by three-regularity condition. This solution is based on the procedure of reduction of the frame bundle attached to the family $\mathbb{B}$. There are two applications of this result. Firstly, in the special case the canonical almost product structure on the family is found. Secondly, in the general case two linear connections are attached intrinsically to the family. The expressions of the curvature tensors of these connections are found. All the considerations are local, all the functions are supposed to be real analytical.

The main concepts

Let P_n be a projective space of dimension n ($n \geq 4$) and P_n^* its dual space.

Definition

A hyperplane element of the space P_n [1] is a pair $x = (L_{n-1}, A)$, where L_{n-1} is a hyperplane in P_n and $A \in L_{n-1}$ is a point. L_{n-1} is called a hyperplane of the element x, and A is a center of x.

Remark. Another terminology is used also: degenerated null-pair by B. Rosenfeld [5], centered $(n-1)$-plane [9], element of a hyperplane [7].

Let $\mathbb{G}^*$ be the set of all hyperplane elements of P_n. The Grassman manifold of all m-dimensional planes in P_n we denote by $\mathbb{G}(m, n)$.

For any subset $B \subset \mathbb{G}^*$ the following maps are defined:

$$\xi\colon B \ni (L_{n-1}, A) \mapsto L_{n-1} \in P_n^*; \quad \eta\colon B \ni (L_{n-1}, A) \mapsto A \in P_n.$$

Definition

A family $\mathbb{B}$ is a smooth $(n-2)$-dimensional family of hyperplane elements satisfying the following conditions:
1) $\eta(\mathbb{B})$ is a smooth p-dimensional surface S_p $(p < n-2)$;
2) the projection $\eta : \mathbb{B} \to S_p$ is a fiber bundle over S_p with $(n-p-2)$-dimensional fibers;
3) $T_A(S_p) \subset L_{n-1}$ for any element $(L_{n-1}, A) \in \eta^{-1}(A)$.

We call the surface $S_p = \eta(\mathbb{B})$ *a base surface* of the family $\mathbb{B}$. The sections $\gamma : S_p \to \mathbb{B}$ of the fiber bundle $\eta : \mathbb{B} \to S_p$ are hyperbands with the same base surface (on hyperbands see [6]).

Definition

A composite clothing of a family $\mathbb{B}$ is a smooth map

$$\mathcal{C} : \mathbb{B} \to \mathbb{G}(p-1,\, n) \times \mathbb{G}(n-p-2,\, n) \times P_n,$$

taking into correspondence to each its element x a set $(N_{p-1}(x),\, N_{n-p-2}(x),\, C_0(x))$ of planes such that

$$N_{p-1}(x) \oplus A = T_A(S_p), \quad T_A(S_p) \oplus N_{n-p-2}(x) = \xi(x),$$

$$\xi(x) \oplus C_0(x) = P_n,$$

where $A = \eta(x)$, $dim N_{p-1}(x) = p-1$, $dim N_{n-p-2}(x) = n-p-2$ and $C_0(x)$ is a point.

The method of solving the problem.

1. Initial choice of a frame bundle over $\mathbb{B}$.

Let $GP(n)$ be the group of projective transformations of P_n and G be the stationary subgroup of a hyperplane element. The family $\mathbb{B}$ is a manifold immersed into $\mathbb{G}^*$. The set of all the frames of $\mathbb{G}^*$ is identified with the manifold $\mathcal{F}(P_n)$ of all projective frames of P_n. A $GP(n)$-invariant diffeomorphism $\mathcal{F}(P_n) \cong GP(n)$ can be given by choosing a frame $\mathcal{R}_0$ correspondent to the unit of $GP(n)$. The projection $\zeta : \mathcal{F}(P_n) \to \mathbb{G}^*$ acts as follows:

$$\zeta : \{A_0, \ldots, A_{n-1}, A_n\} \mapsto (\langle A_0, \ldots, A_{n-1} \rangle, A_0).$$

Then $\mathcal{F}(\mathbb{B}) = \zeta^{-1}(\mathbb{B})$ is equipped by the structure of principal bundle over $\mathbb{B}$ with the structure group G. This bundle is called *a zero order frame bundle* on $\mathbb{B}$.

2. An analytical description of the frame bundle in terms of differential forms

Due to the Frobenius theorem, the manifold $\mathcal{F}(\mathbb{B})$ can be given analytically as a fiber of involutive distribution $\mathcal{D}$ on $\mathcal{F}(P_n)$ defined by some Pfaffian system of equations in terms of the left-invariant forms of the group $GP(n)$.

They express fiber forms in terms of base ones of the family $\mathcal{F}(\mathbb{B})$ where the fiber forms and the base forms are defined as follows. In any chart (x^i, y^α) on $\mathcal{F}(\mathbb{B})$ the base forms are expressed in terms of base point coordinates (x^i) while the fiber forms are expressed in terms of both x^i and y^α.

The coefficients in the equations are functions on $\mathcal{F}(\mathbb{B})$. Taken together, they determine some field of geometric object Λ called *the field of the first order fundamental object of* $\mathbb{B}$. Thus the involutivity condition is represented by the Pfaffian equations imposed on the components of Λ and its prolongations.

3. A stepwise reduction of the frame bundle.

We make the reduction until the orbits of the frame points in the fiber over any $x \in \mathbb{B}$ determine the planes of some composite clothing.
The bundle can be reduced either by geometrical or by analytical ways. The geometrical way is distinguishing subset of frames whose vertices are on algebraic varieties intrinsically attached to an element of $\mathbb{B}$ (see, e.g. [8]). The analytical one is based on the Ostianu lemma [4].

The Ostianu lemma

Let M be a smooth m-dimensional manifold immersed into a representation space of a Lie group. Let a geometric object field X on M is determined by the Pfaffian system of equations

$$dX^\alpha + \xi^\alpha_a(X^\beta)\omega^a = X^\alpha_k \omega^k \quad (\alpha, \beta = \overline{1, p},\ a = \overline{1, r},\ k = \overline{1, m}),\ (*)$$

where ω^k are the principal forms and ω^a are the secondary ones. Let us assign some constant values to all the X^α's. Suppose that the system (*) contains some sub-system of σ equations from which σ secondary forms can be expressed. Let X^{α_0} be those components of X which differentials are in this subsystem. Then there exists a frame subbundle over M such that the components X^{α_0} preserve the constant values assigned above.

We consider a moving frame $\{A_0, A_I\}$ $(I, J, \ldots = \overline{1, n})$ in P_n with the formulas determining infinitesimal displacement of the frame:

$$dA_0 = \theta A_0 + \omega^I A_I, \quad dA_I = \theta A_I + \omega_I^J A_J + \omega_I A_0, \tag{1}$$

where the Maurer – Cartan forms ω^I, ω^I_J, ω_I of the projective group $GP(n)$ satisfy the equations

$$\begin{gathered} d\omega^I = \omega^J \wedge \omega^I_J, \quad d\omega_I = \omega_I^J \wedge \omega_J, \\ d\omega^I_J = \omega^K_J \wedge \omega^I_K + \omega^J \wedge \omega_I + \delta^I_J \omega_K \wedge \omega^K. \end{gathered} \tag{2}$$

Any frame $\{A_0, A_i, A_u, A_y, A_n\}$, belonging to the fiber of $\mathcal{F}(\mathbb{B})$ over $x = (L_{n-1}, A)$, satisfies the following conditions

$$A_0 = A, \quad A_i \in T_A(S_p), \quad A_u,\, A_y \in L_{n-1},$$

$$i, j, \ldots = \overline{1, p}; \quad u, v, \ldots = \overline{p+1, n-2}; \quad y = n-1.$$

The equations of the $\mathcal{F}(\mathbb{B})$ are following:

$$\omega^u = 0, \quad \omega^y = 0, \quad \omega^n = 0, \quad \omega_i^u = \Lambda_{ij}^u \theta^j, \quad \omega_i^y = \Lambda_{ij}^y \theta^j,$$

$$\omega_i^n = \Lambda_{ij}^n \theta^j, \quad \omega_y^n = \Lambda_{yi}^n \theta^i + \Lambda_{yn}^{nu} \theta_u^n, \tag{3}$$

$$\Lambda_{ij}^u = \Lambda_{ji}^u, \quad \Lambda_{ij}^y = \Lambda_{ji}^y, \quad \Lambda_{ij}^n = \Lambda_{ji}^n,$$

where $\theta^i = \omega^i$, $\theta_u^n = \omega_u^n$ are the base forms,

$$\Lambda_1 = \{\Lambda_{ij}^u, \Lambda_{ij}^y, \Lambda_{ij}^n, \Lambda_{yi}^n, \Lambda_{yn}^{nu}\}$$

is the first order fundamental object of $\mathbb{B}$.

The exterior differentials on the base forms $\theta^i = \omega^i$, $\theta^n_u = \omega^n_u$:

$$d\theta^i = \theta^j \wedge \omega^i_j, \quad d\theta^n_u = \theta^i \wedge \Theta^n_{ui} + \theta^n_v \wedge \Theta^{nv}_{un}, \tag{4}$$

where $\Theta^n_{ui} = -\Lambda^n_{ij}\omega^j_u - \Lambda^n_{yi}\omega^y_u, \quad \Theta^{nv}_{uN} = -\omega^v_u + \delta^v_u\omega^n_n - \Lambda^{nv}_{yn}\omega^y_u.$
(4) $\Longrightarrow$ complete integrability of the following Pfaffian systems:

$$1)\, \theta^i = 0, \quad 2)\, \theta^i = 0, \ \ \theta^n_u = 0.$$

By "$\equiv$" we denote comparisons modulo the base forms θ^i, θ^n_u:

$$\omega \equiv 0 \quad \Longleftrightarrow \quad \omega = (\ldots)_i\theta^i + (\ldots)^u_n\theta^n_u.$$

Equations on the components of Λ_1 can be written as follows:

$$\begin{gathered} \Delta\Lambda^u_{ij} + \Lambda^n_{ij}\omega^y_n + \Lambda^n_{ij}\omega^u_n \equiv 0, \\ \Delta\Lambda^y_{ij} + \Lambda^u_{ij}\omega^y_u + \Lambda^n_{ij}\omega^y_n \equiv 0, \quad \Delta\Lambda^n_{ij} \equiv 0, \\ \Delta\Lambda^{nu}_{yn} + \Lambda^{nv}_{yn}\Lambda^{nu}_{yn}\omega^y_v - \omega^u_y \equiv 0, \\ \Delta\Lambda^n_{yi} - \Lambda^n_{ij}\omega^j_y + \Lambda^{nu}_{yn}\Theta^n_{ui} \equiv 0, \end{gathered} \tag{5}$$

where, for example, $\Delta\Lambda^n_{yi} = d\Lambda^n_{yi} + \Lambda^n_{yi}\omega^n_n - \Lambda^n_{yi}\omega^y_y - \Lambda^n_{yj}\omega^j_i$.

Definition

A family $\mathbb{B}$ is said to be regular if the tensor field $\{\Lambda^n_{ij}\}$ is non-degenerate, i.e.

$$a_1 = det\|\Lambda^n_{ij}\| \neq 0. \qquad (6)$$

Then we can introduce the inverse tensor V^{ij}_n:

$$V^{ik}_n \Lambda^n_{kj} = \delta^i_j,$$

and the following objects:

$$\lambda^u_n = \frac{1}{p}\Lambda^u_{jk} V^{jk}_n, \quad \lambda^y_n = \frac{1}{p}\Lambda^y_{jk} V^{jk}_n. \qquad (7)$$

Reduction of the frame bundle $\mathcal{F}(\mathbb{B})$

Step 1.

$$\omega_y^n = 0 \overset{(3)}{\Longleftrightarrow} \Lambda_{yi}^n = 0 \,\&\, \Lambda_{yn}^{nu} = 0 \overset{(5)(6)}{\Longrightarrow} \omega_y^i \equiv 0 \,\&\, \omega_y^u \equiv 0. \qquad (8)$$

$$\omega_y^i = \Lambda_{yj}^i \theta^j + \Lambda_{yn}^{iu} \theta_u^n, \quad \omega_y^u = \Lambda_{yi}^u \theta^i + \Lambda_{yn}^{uv} \theta_v^n.$$

$$\begin{gathered} \Delta\Lambda_{yi}^u - \Lambda_{yn}^{uv} \Theta_{vi}^n \equiv 0, \quad \Delta\Lambda_{yn}^{uv} \equiv 0, \\ \Delta\Lambda_{yj}^i + \Lambda_{yj}^u \omega_u^i - \Lambda_{yn}^{iu} \Theta_{uj}^n - \delta_j^i \omega_y \equiv 0, \\ \Delta\Lambda_{yn}^{iu} + \Lambda_{yn}^{vu} \omega_v^i \equiv 0, \\ \Delta\lambda_n^u + \omega_n^u \equiv 0, \quad \Delta\lambda_n^y + \lambda_n^u \omega_u^y + \omega_n^y \equiv 0, \\ \Delta\Lambda_{ij}^n = \Lambda_{ijk}^n \theta^k - \Lambda_{ij}^u \theta_u^n. \end{gathered} \qquad (9)$$

Step 2.

$$\lambda_n^u = 0 \overset{(9)}{\Longrightarrow} \omega_n^u \equiv 0 \Longrightarrow \omega_n^u = \lambda_{ni}^u \theta^i + \lambda_{nn}^{uv} \theta_v^n,$$
$$\lambda_n^y = 0 \overset{(9)}{\Longrightarrow} \omega_n^y \equiv 0 \Longrightarrow \omega_n^y = \lambda_{ni}^y \theta^i + \lambda_{nn}^{yv} \theta_v^n. \qquad (10)$$

$$\Delta\lambda_{ni}^u + \Lambda_{ji}^u \omega_n^j \equiv 0, \quad \Delta\lambda_{ni}^y - \Lambda_{ji}^y \omega_n^j + \lambda_{ni}^u \omega_u^y \equiv 0,$$
$$\Delta\lambda_{nn}^{uv} \equiv 0, \quad \Delta\lambda_{nn}^{yu} + \lambda_{nn}^{vu} \omega_v^y \equiv 0. \qquad (11)$$

Definition

A regular family $\mathbb{B}$ is said to be 2-regular if the tensor field $\{\lambda_{nn}^{uv}\}$ is non-degenerate, i.e.

$$a_2 = det\|\lambda_{nn}^{uv}\| \neq 0. \qquad (12)$$

Step 3.

$$\lambda^{yu}_{nn} = 0 \overset{(11)}{\Longrightarrow} \lambda^{vu}_{nn}\omega^y_v \equiv 0 \overset{(12)}{\Longrightarrow} \omega^y_u \equiv 0. \tag{13}$$

$$(5) \overset{(10)(13)}{\Longrightarrow} \Delta\Lambda^y_{ij} \equiv 0,$$
$$(11) \overset{(13)}{\Longrightarrow} \Delta\lambda^y_{ni} - \Lambda^y_{ji}\omega^j_n \equiv 0. \tag{14}$$

Definition

A 2-regular family $\mathbb{B}$ is said to be 3-regular if the tensor field $\{\Lambda^y_{ij}\}$ is non-degenerate, i.e.

$$a_3 = det\|\Lambda^y_{ij}\| \neq 0. \tag{15}$$

Step 4.

$$\lambda^y_{ni} = 0 \overset{(14)(15)}{\Longrightarrow} \omega^i_n \equiv 0 \Longrightarrow \omega^i_n = \lambda^i_{nj}\theta^j + \lambda^{iu}_{nn}\theta^n_u. \tag{16}$$

$$\begin{gathered} \Delta\lambda^i_{nj} + \lambda^u_{nj}\omega^i_u - \lambda^{iu}_{nn}\Lambda^n_{jk}\omega^k_u - \delta^i_j\omega_n \equiv 0, \\ \Delta\lambda^{iu}_{nn} + \lambda^{vu}_{nn}\omega^i_v \equiv 0. \end{gathered} \tag{17}$$

Step 5.

$$\Lambda^{iu}_{nn} = 0 \overset{(12)(17)}{\Longrightarrow} \omega^i_u \equiv 0 \Longrightarrow \omega^i_u = \lambda^i_{uj}\theta^j + \lambda^{iv}_{un}\theta^n_v. \tag{18}$$

$$\Delta\lambda^i_{uj} - \delta^i_j\omega_u \equiv 0, \quad \Delta\lambda^{iv}_{un} \equiv 0. \tag{19}$$

$$(9) \overset{(13)(18)}{\Longrightarrow} \Delta\Lambda^i_{yj} - \delta^i_j \omega_y \equiv 0, \quad (17) \overset{(18)}{\Longrightarrow} \Delta\Lambda^i_{nj} - \delta^i_j \omega_n \equiv 0. \tag{20}$$

Let us consider the objects

$$\mu_u = \Lambda^i_{ui}, \quad \mu_y = \Lambda^i_{yi}, \quad \mu_n = \Lambda^i_{ni}, \quad \mu_k = \Lambda^n_{ijk} V^{ij}_n. \tag{21}$$

$$\begin{aligned}
(21) &\overset{(19)}{\Longrightarrow} \Delta\mu_u - p\omega_u \equiv 0, \\
(21) &\overset{(20)}{\Longrightarrow} \Delta\mu_y - p\omega_y \equiv 0, \\
(21) &\overset{(20)}{\Longrightarrow} \Delta\mu_n - p\omega_n \equiv 0, \\
(21) &\overset{(9)}{\Longrightarrow} \Delta\mu_i + (p+2)\omega_i \equiv 0.
\end{aligned} \tag{22}$$

Step 6.

$$\begin{array}{cccc} \mu_u = 0 & \mu_y = 0 & \mu_n = 0 & \mu_i = 0 \\ (22) \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \omega_u \equiv 0 & \omega_y \equiv 0 & \omega_n \equiv 0 & \omega_i \equiv 0. \end{array} \qquad (23)$$

From (1), (3), (8), (10), (13), (16) and (23) it follows that all the Maurer – Cartan forms except ω^i_j, ω^u_v, ω^y_y, ω^n_n are expressed by the base forms. Then the following spans of the vertices of the adapted frame become invariant if $x \in \mathbb{B}$ is fixed:

$$N_{p-1} = \langle A_1, \dots, A_p \rangle, \; N_{q-1} = \langle A_{p+1}, \dots, A_{n-1} \rangle, \; C_0 = A_n. \quad (24)$$

$$\Downarrow$$

Theorem (A.Kuleshov, 2014)

The composite clothing consisting of the plane fields (24) is attached intrinsically to a three-regular family $\mathbb{B}$.

Geometrical meaning of the regularity condition

We denote by $E(\mathbb{B})$ the enveloping surface of the smooth family $\xi(\mathbb{B})$ of hyperplanes.

Definition

A characteristic $F_r(x)$ of $\mathbb{B}$ at x is the plane generator of $E(\mathbb{B})$ at x.

Theorem

A family $\mathbb{B}$ is regular iff for any $x \in \mathbb{B}$ we have $F_r(x) \cap T_A(S_p) = A$, where $A = \eta(x)$.

Remark. It the case of regular family we have $dimF_r(x) = 1$.

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