Bifurcations in 3 d.o.f. integrable Hamiltonian systems

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Outline

Integrable Hamiltonian systems (IHS) are nowadays a rather elaborated part of the dynamical theory of Hamiltonian systems. This study is useful in many aspects, one of them is the understanding of their structure for the purposes of the perturbation theory: an integrable system is the starting point to understand the structure of the perturbed system. Also the study of the related Liouville foliation has its own value for the geometry. When one deals with a Hamiltonian system of which is known as being integrable, one may distinguish several aspects of its study. The primary aspect is about the local structure of the system near some its simplest singular sets: equilibria, periodic orbits and invariant tori. Main results in this direction are due to contributions of Birkhoff , Moser, Rüssmann, Vey, Eliasson, Ito, Zung, Miranda for analytic and C^∞ systems. Some local results concerning finitely smooth integrable systems are due to L-U. Semi-local aspects consist in the study of a given IHS in some neighborhoods of its singular sets saturated w.r.t. to the orbits of the induced Poisson action. This was initiated by Lerman-Umanskiy (1981) and a bit later Fomenko and continued in his school: Bolsinov, Zung, Oshemkov, Kalashnikov, Matveev, and many others.

Here the topic of bifurcations in IHS arises very naturally. First of all, it is a common case when integrable systems are met in families, and all members of the family are IHS. But the structure of integrable systems themselves give rise to bifurcations. To explain this, recall that in Hamiltonian systems singular orbits like periodic orbits, invariant tori are met in families of different dimension in the phase space. For instance, periodic orbits usually belong to 1-parameter families forming 2-dim submanifold (this is valid not only for an integrable system but for any system with the first integral), invariant k-dimensional isotropic tori usually belong to k-parameter families forming generically $2k$-dimensional submanifolds. The most important property of an integrable system is that near such submanifolds the system can be reduced by means of some reduction procedure to the family of IHS of the lesser dimension but depending on parameters. The experience of different branches of the theory of dynamical systems, singularity theory of smooth functions, etc., suggest: the more parameters a system has the more degenerate singularity can be met in the family. Below we consider only the case $n = 3$.

Let a C^∞-smooth 3D IHS $(X_H,\ F_1,\ F_2)$ be given. We assume that X_H has an one-parameter family of periodic trajectories γ_c being one-dimensional orbits of the related Poisson action. Let m be a point on some γ_c. The common reduction procedure gives us a family of four-dimensional symplectic disks with reduced Hamiltonians with singular points depending on one parameter. This is done as follows. At m differentials $dH,\ dF_1,\ dF_2$ generate a 1-dim subspace in $T^*_m M$. Taking, if necessary, linear combinations of integrals one may suppose $dF_2 \neq 0$ at m. One may choose near m symplectic coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$ in such a way that $F_2 = x_3$ in these coordinates (Nekhoroshev) and $m = (0,0,0,0,0,0)$. Then, due to pair-wise involution for integrals $H, F_2,, F_1, F_2$, functions H, F_1 in these coordinates do not depend on y_3. At m co-vectors dH and dF_1 are collinear to dF_2, therefore differentials dH, dF_1 vanish w.r.t. reduced variables (x_1, x_2, y_1, y_2), one gets a one-parameter family of integrable systems in two degrees of freedom with Hamiltonian H and additional integral F_1, the point $\hat{m}$ (projection of m to the reduced space) is the equilibrium for X_H. The type of this point and its continuation in parameter x_3 depends on the type of singularity of the triple (H, F_1, F_2).

Denote $x = (x_1, x_2)$, $y = (y_1, y_2)$. We assume that this singularity is not too degenerate, namely
Assumption 1. Rank of 4×5 matrix

$$\begin{pmatrix} \frac{\partial}{\partial x_3}(\frac{\partial H}{\partial y}) & \frac{\partial}{\partial x}(\frac{\partial H}{\partial y}) & \frac{\partial}{\partial y}(\frac{\partial H}{\partial y}) \\ \frac{\partial}{\partial x_3}(\frac{\partial H}{\partial x}) & \frac{\partial}{\partial x}(\frac{\partial H}{\partial x}) & \frac{\partial}{\partial y}(\frac{\partial H}{\partial x}) \end{pmatrix} \quad (1)$$

is equal to 4. In this case, the set of 1-dim orbits near m indeed forms a smooth 1-parameter family.
If, in addition, Hessian of H in variables (x, y) does not vanish at $\hat{m}$, then singularity at $\hat{m}$ can be continued in parameter $x_3 = \varepsilon$ and one has an isolated non-degenerate equilibrium for every ε. But such one-parameter family can meet unavoidably degenerate singular points without zero eigenvalues.
Another situation arises, if the Hessian of H is equal to zero at m but a minor containing the column with derivatives in x_3 in matrix (1) does not vanish. Then for $\varepsilon = 0$ the equilibrium at $\hat{m}$ does have at least one double zero eigenvalue generically non-semisimple one.

Now we enumerate changes of singular point types that can be met in generic 1-parameter families of IHS in two degrees of freedom. They are determined first by the transitions of eigenvalues and corresponding Jordan forms at that value of the parameter when a singular point of the vector field degenerates (critical value). Generically, the following is possible

- as a governing parameter varies, among four different imaginary eigenvalues (elliptic singular point) two pairs becomes two imaginary double non-semisimple ones, and then turn out into a complex quadruple (usually such the bifurcation is called as the Hamiltonian Hopf Bifurcation (Meer), if some coefficient in nonlinear normal form does not vanish);
- when changing a parameter, four different real eigenvalues (saddle singular point) become two double non-semisimple real nonzero eigenvalues and then turn out to be a complex quadruple;

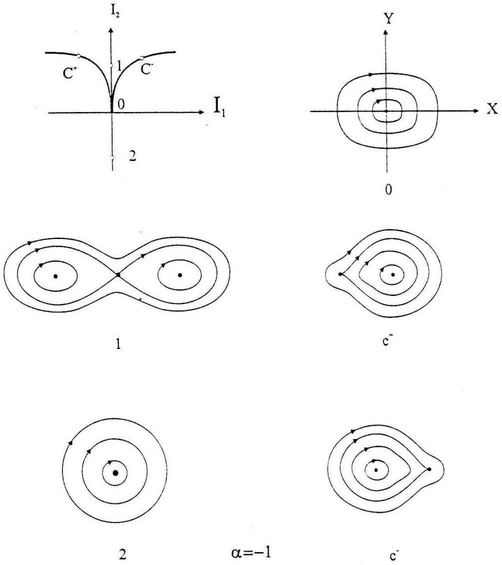
- when changing a parameter, among four different imaginary eigenvalues (again an elliptic singular point) one pair collides and becomes double zero non-semisimple one, and then one gets two imaginary eigenvalues and two reals (such a bifurcation is called sometimes as the elliptic Hamiltonian Hopf Bifurcation (B-C-K-V), if some coefficient in the nonlinear normal form does not vanish);
- when changing a parameter among four different real eigenvalues (saddle singular point) one pair becomes double non-semisimple zero and then eigenvalues turn out to be a pair of reals and a pair of imaginary eigenvalues (center-saddle singular point).

First two bifurcations are well studied, therefore we only briefly describe results and pictures (see details and pictures in (Meer,AKN) and codimension 2 case in (G-L) for the Hamiltonian Hopf bifurcation, and (AKN,BCKV) for normally elliptic bifurcation). But the local study shows the existence of cases when separatrix sets of the singular points go out the neighborhood of the singular point, then one needs to add semi-local description.

If m is a point on a 2-dimensional invariant torus, then the reduction procedure leads to the a family of 1 d.o.f. Hamiltonian systems depending on two parameters $(\varepsilon_1, \varepsilon_2)$ (values of two independent integrals, say F_1, F_2). This means that such a family can generically contain a system with a singular point of codimension 2, in this case it can be either a degenerate elliptic point or degenerate saddle. Such a point has as its linearization matrix the double zero eigenvalue with 2-dimensional Jordan box, zero coefficient in the normal form the third order for the critical Hamiltonian and nonzero coefficient in the normal form of the fourth order. This normal form with parameters $\varepsilon_1, \varepsilon_2$ is as follows

$$H = a(\varepsilon_1, \varepsilon_2)x + \frac{1}{2}y^2 + \frac{b(\varepsilon_1, \varepsilon_2)}{2}x^3 + \frac{c(\varepsilon_1, \varepsilon_2)}{2}x^4 + \cdots ,$$

here $a(0,0) = b(0,0) = 0$, $c(0,0) \neq 0$, $D(a,b)/D(a(\varepsilon_1, \varepsilon_2)) \neq 0$. Related phase portraits are plotted in Figs.3-4 for both signs of $c(0,0)$.


I2
1
C+
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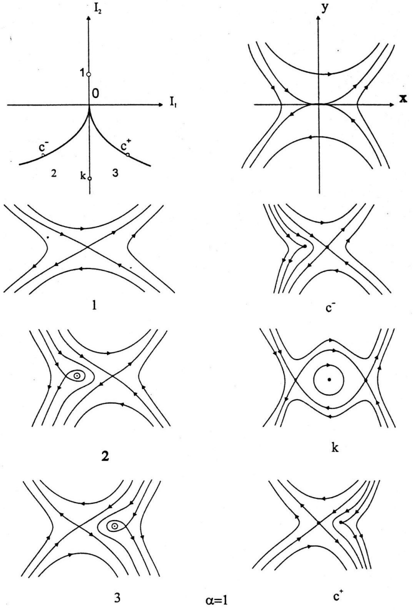
I_2
1
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3
y
x
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c^-
2
k
3
$\alpha=1$
c^+

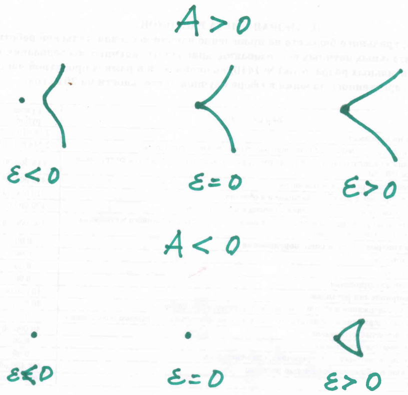
At the critical value of a governing parameter (say ε) a system has two double pure imaginary eigenvalues $\pm i\omega$ both non semi-simple ones. The related normal form (Sokol'sky, Meer) looks as follows

$$H_\varepsilon = \frac{1}{2}(y_1^2 + y_2^2) + \omega(x_1 y_2 - x_2 y_1) - \frac{\nu(\varepsilon)}{2}(x_1^2 + x_2^2) +$$

$$(x_1^2 + x_2^2)[A(x_1^2 + x_2^2) + B(x_1 y_2 - x_2 y_1) + C(y_1^2 + y_2^2)] + \sum_{\alpha_1+\alpha_2+\alpha_3=3}^{\infty} h_{\alpha_1,\alpha_2,\alpha_3}(x_1^2 + x_2^2)^{\alpha_1}(x_1 y_2 - x_2 y_1)^{\alpha_2}(y_1^2 + y_2^2)^{\alpha_3}, \tag{2}$$

One assumes coefficient A do not vanish. The type of bifurcation depends of the sign of A. For A positive the bifurcation is pure local, for A negative it is semi-local. Bifurcation diagrams for both cases are plotted in Fig.1 and related phase portraits of the reduced systems are on Fig.2.

For the case $A < 0$ separatrix sets for the focus-focus and degenerate elliptic point leave a neighborhood of the equilibrium. So, a semi-local consideration is necessary to get a saturated neighborhood of the equilibrium. This does not change the bifurcation set.


$A > 0$
$\varepsilon < 0$
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$A < 0$
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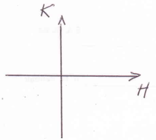
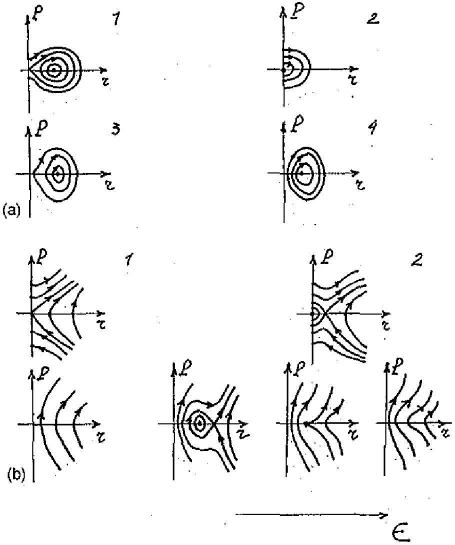

K
H



(a)
(b)

Let us assume that linearizarion matrix for X_{H_0} at p_0 has two double non semi-simple eigenvalues $\lambda, -\lambda$ (their related Jordan boxes are 2-dimensional). Then the quadratic part of the Hamiltonian can be reduced to the following Williamson normal form (Galin, Burgoyne-Cushman, Kocak)

$$H_0^{(2)} = \lambda(x_1y_1 + x_2y_2) + x_1y_2,\ \lambda > 0.$$

A generic 1-parameter unfolding of this quadratic Hamiltonian can be reduced (after some rescaling the parameter) to the following family

$$H_\varepsilon^{(2)} = \lambda(\varepsilon)(x_1y_1 + x_2y_2) + x_1y_2 + \varepsilon x_2y_1.$$

Supposing the Hamiltonian to be real analytic or C^∞-smooth, one can perform the local normalization procedure up to any desirable order. Following the lines of (Sokol'sky, Meer, Chow) we come to the following 4th order normal form

$$H_\epsilon = \lambda(\varepsilon)(x_1y_1+x_2y_2)+x_1y_2+\varepsilon x_2y_1+Ax_2^2y_1^2+Bx_2y_1(x_1y_1+x_2y_2)+C(x_1y_1+x_2y_2) \quad (3)$$

here A, B, C depend smoothly on ε.

Assumption 2. $A(0) \neq 0$.

Then the equilibrium p_0 is a degenerate saddle of X_{H_0}, equilibria $p_\varepsilon,\ \varepsilon < 0$, are focus-foci, and $p_\varepsilon,\ \varepsilon > 0$, are saddle-saddles (we follow here the terminology of L-U for integrable systems). This implies, in particular, that one can assume p_ε unmoved, $p_\varepsilon = p$ (shifting coordinates, if necessary) that is supposed later on.

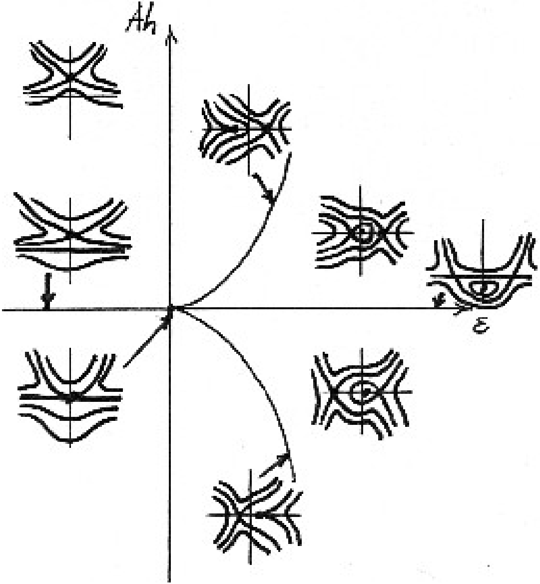
In principle, the singular leaf of point p, i.e. joint connected leaf of two integrals H, F can contain other singular points. To make the exposition more transparent we assume below

Assumption 3. For any small ε a singular leaf $H_\varepsilon = 0$, $F_\varepsilon = F_\varepsilon(p) = 0$ containing p is compact and has the only singular point p and does not contain closed one-dimensional orbits.

Let us consider first the local orbit structure of the corresponding vector field and Poisson action near p for ε small enough. The local structure of an integrable system is completely determined by its normal form of any order, we restrict ourself by the normal form of the fourth order (see (3) without dots) which captures all needed properties of the system. As an additional local integral we take $Q = x_1y_1 + x_2y_2$. Since p is of the saddle type for all ε small enough, there are two smooth local mutually transverse Lagrangian disks through p, namely its stable $W^s : y_1 = y_2 = 0$ and unstable $W^u : x_1 = x_2 = 0$ manifolds. These manifolds belong to the singular leaf containing p. But solutions of the system $H = Q = 0$ are not exhausted with the points of stable and unstable disks, other solutions exist. Other points lie on the graph defined by functions

$$y_1 = \frac{x_1^2 - \varepsilon x_2^2}{Ax_2^3},\ y_2 = -\frac{x_1(x_1^2 - \varepsilon x_2^2)}{Ax_2^4}, \tag{4}$$

if $x_2 \neq 0$.


Ah
ε

To understand the Liouville foliation structure near loops and its reconstructions upon changing parameter ε, we choose, as was said, two cross-sections N^s_+, N^s_- to ingoing parts of Γ_i and two others $N^u_+ : y_1 = d$, $N^u_- : y_1 = -d$ to outgoing parts of Γ_i. Every cross-section is foliated by levels $H_\varepsilon = h$ into smooth two-dimensional curvilinear rectangles given for N^s_+ and N^s_- by inequalities $||y|| \le d/2,\ |x_1| \le \delta$ with (symplectic) coordinates (x_1, y_1) on them and for N^u_+ and N^u_- by inequalities $||x|| \le d/2,\ |y_2| \le \delta$ with (symplectic) coordinates (x_2, y_2) on them. Each rectangle is a graph defined by equations $H_\varepsilon = h,\ x_2 = d,$ or $x_2 = -d$ and similar for two other cross-sections. The Liouville foliation is transverse to the cross-section, so it generates a one-dimensional foliation on each rectangle, this foliation is given by level lines of the restriction of the function Q on the rectangle (though Q does not depend on ε but the rectangle itself depends on parameter ε, therefore the foliation will depend on two parameters h, ε).

In order to describe the structure of Liouville foliation within a neighborhood of p, we need to understand how level lines generated by function Q on a cross-section are transformed by X_{H_ε}-flow when its trajectories pass inside a neighborhood of p till they reach cross-sections N^u_+ or N^u_-. The first question here is about traces of the joint level $H_\varepsilon = Q = 0$ where p lies. At $\varepsilon = 0$ the curve (**??**) intersects the trace of W^s at only one point, therefore, since p is a saddle, through its points other than $(0,0,0)$ pass trajectories which stay in the neighborhood of p only a finite time and then leave this neighborhood (though the closer this point to $(0,0,0)$ the longer a respective time is). To be definite, we assume $A > 0$ in the following considerations.

Lemma

*At $\varepsilon = 0$ and $x_1 \neq 0$ the curve (**??**) is mapped onto $N^u_+ : y_1 = d$, and the related curve from N^s_- with $x_1 \neq 0$ is mapped on $N^u_- : y_1 = -d$. For small positive ε the part of the curve with $y_1 > 0$ is mapped to N^u_+ and that with $y_1 < 0$ is mapped to N^u_-. For ε small negative all the curve with sufficiently small x_1 is mapped onto N^u_+.*

Here we consider a one parameter integrable unfolding of an integrable Hamiltonian system in 2 degrees of freedom that has a singular point p with its linearization having one double non semi-simple zero eigenvalue and a pair of real simple eigenvalues $\pm\lambda$. Near p the Hamiltonian at the critical value of the parameter (say, $\varepsilon = 0$) can be written in some symplectic coordinates (x, y, u, v) in the form (Galin)

$$H_2 = \lambda xy \pm v^2/2 + \cdots,\ \Omega = dx \wedge dy + du \wedge dv,$$

here dots mean the terms of the order 3 and higher, λ can suppose to be positive. Such Hamiltonian vector field X_H has near p a local smooth invariant symplectic 2-dimensional center manifold W^c (2-disk) corresponding to the double zero eigenvalue. The restriction of X_H onto W^c is a one-degree-of-freedom Hamiltonian vector field with a parabolic equilibrium at p, if some coefficient in the normal form of the third order in the Hamiltonian does not vanish (its quadratic part is $\pm v^2/2$).

All trajectories on W^c, except for the point p, are 1-dim orbits of the induced Poisson action, the point p is 0-dim action orbit. Submanifold W^c is a local hyperbolic submanifold of X_H in the sense of Hirsh-Pugh-Shub and Fenichel. The equilibrium p possesses also strong stable W^{ss} and strong unstable W^{uu} local manifolds being smooth 1-dim curves through p, they correspond to eigenvalues $\mp\lambda$, respectively. They also form 1-dim orbits (except for the point p) of the Poisson action. But a local stable (unstable) set for p, that is the set of all positive (negative) semi-trajectories which tend to p as $t \to \infty$ ($t \to -\infty$), are not exhausted by its strong stable (unstable) curves, since parabolic point on W^c also has stable and unstable curves (each of them is a local segment with p being its extreme point, their union compose the curve like semi-cubic parabola with the cusp at p). The following assertion is valid

Proposition. The set of all local semi-trajectories of X_H tending to p as $t \to \infty$ makes up the set W^s diffeomorphic to semi-disk on the plane (x, y): $x \geq 0$, $x^2 + y^2 < 1$. The same is true for the set W^u of all local semi-trajectories of X_H tending to p as $t \to -\infty$. These two sets locally near p intersect each other at only point p (Fig.).

For the integrable Hamiltonian vector field these sets W^s, W^u are locally only the part of the solution set for the system $H = H(p)$, $F = F(p)$, where F is the second smooth integral near p. This set is diffeomorphic to the direct product of the set $H = H(p)$ on W^c (semi-cubic parabola) and the cross $xy = 0$ on the plane (x, y). Below we assume
Assumption 4. In the phase space a connected component of the whole level set containing p given by the equations $H = H(p)$, $F = F(p)$ is compact and has the only singular point p (0-dim orbit) of the related Poisson action.
Locally near p there are exactly six 1-dim action orbits adjacent to p: two of them belong to W^{uu}, two others do to W^{ss} and two more do to semi-cubic parabola. The local structure of the action near every of these 1-dim orbits can be studied by means of the reduction procedure described above. This leads to the following statement.
Proposition. For a point m on strong stable (unstable) curve the reduction gives a 1 d.o.f. Hamiltonian system with a parabolic singular point. For a point m on the stable (unstable) curve for p on the center manifold (being 1-dim action orbits) the reduction gives a 1 d.o.f. Hamiltonian system with a saddle singular point.

This proposition allows us to conclude that the local structure of the solution set of the system $H_0 = F = 0$ near a point on W^{ss} or W^{uu} is diffeomorphic to the direct product of the cuspoidal curve and a segment. For the point on a separatrix (stable or unstable) of a parabolic point on W^c the local structure of this set is diffeomorphic to the direct product of a cross $xy = 0$ on the (x, y)-plane near the origin and a segment.

Theorem. For a C^∞-smooth Hamiltonian vector field in two degrees of freedom with a singular point with two simple real eigenvalues $\pm\lambda$ and double non semi-simple zero eigenvalue there is a smooth symplectic coordinates (x, y, u, v) near p, $\Omega = dx \wedge dy + du \wedge dv$, such that in these coordinates the Hamiltonian takes the following finite order polynomial normal form

$$H = \lambda xy \pm v^2/2 + P(xy, u) + \cdots, \ Q = xy. \quad (5)$$

where $P(\xi, u)$ is a polynomial of some order and dots mean higher order terms. The normalization can be carried out up to any given order.

In order to get the codimension 1 case, we impose the following non-degenericity condition

Assumption 5. The coefficients a, b in P in front of terms xyu and u^3, respectively, do not vanish. In particular, this guarantees the singular point on the center manifold $x = y = 0$ near p to be of the parabolic type.

To obtain all conclusions about local structure it is sufficient to work with the normal form of the third order: $H_3 = \lambda xy \pm v^2/2 + axyu + bu^3$, $a, b \neq 0$.

Now we extend orbits out of a neighborhood of p. Let us make more precise the structure of the set given by equations $H = Q = 0$ in a small neighborhood of p. It consists of four 2-dim semi-disks W^s, W^u, S_1, S_2 invariant w.r.t. X_H-flow. These four semi-disks are divided into two pairs: W^s, S_1, and W^u, S_2, two first ones have common boundary segment W^{ss} and two others have common segment W^{uu}. The behavior of trajectories on S_1, S_2 are of a saddle type: S_1 contains boundary segment W^{ss} and also the unstable separatrix of the semi-cubic parabola, for S_2 the boundary segment is W^{uu} and it also contains the stable separatrix of the semi-cubic parabola.

Semi-disks W^s, W^u, intersect at only point p, the same true for S_1, S_2, but W^s and S_2 intersect along stable separatrix of the semi-cubic parabola, and W^u and S_1 intersect along unstable separatrix of the semi-cubic parabola. It is necessary to notice that each semi-disk of four mentioned consists of two 2-dim action orbits, three 1-dim action orbits (separatrices) and 0-dim orbit – the point p itself. W^{uu} can be continued by X_H-flow for all $\mathbb{R}$ due to compactness of the level component of p and thus the boundary set of extended W^{uu} is a singular point, that is p. Since we extend the orbit by the X_H-flow, the extended trajectory (one of two existing) from W^{uu} has to merge with one of two trajectories in W^{ss} (and not with trajectories from semi-cubic parabola). It follows from the local structure of the set $H = Q = 0$ near p. Thus we get a homoclinic orbit to p. The same is true, if we extend the second trajectory in W^{uu}, so we get two homoclinic orbits γ_1, γ_2 to p forming together a "figure eight". One more homoclinic orbit Γ is formed, when one extends the unstable curve of the semi-cubic parabola which has to merge under extending with the stable curve of the same parabola. When extending them around Γ, we get a symplectic annulus containing Γ whose other orbits are closed 1-dim

Bifurcations in a one-parameter unfolding

What can be said about the orbit structure within the levels $H = c$ for small positive $|c|$? As an unfolding of the degenerate system we consider an integrable system which in local symplectic coordinates (x, y, u, v) is of the form

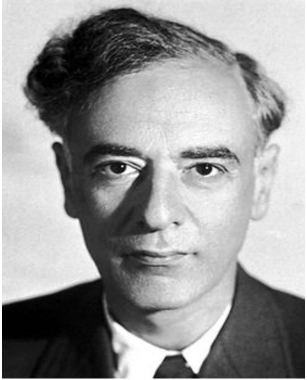
$$H = \varepsilon u + \lambda xy \pm v^2/2 + P(xy, u) + \cdots , \quad (6)$$

where coefficients in the polynomial P now depend on parameter ε. The global part of the system depends smoothly on ε. The local changes of the structure are displayed in the change of the structure on the center submanifold. This submanifold stays to be hyperbolic in the transverse directions.
The known bifurcations of this 1 d.o.f. Hamiltonian system are: 1) the break of the parabolic point into center and saddle equilibria, two separatrices (one stable and one unstable) of the saddle merge forming one homoclinic loop for $\varepsilon > 0$ (to be definite), this homoclinic loop is the boundary of a neighborhood of the center filled with periodic orbits, and 2) disappearance of equilibria on the center manifold, all

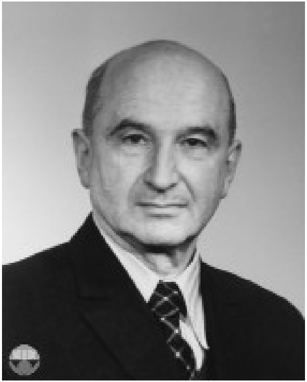
In the whole phase space the equilibria appeared at positive ε on the center manifold are saddle-center and a saddle-saddle. All periodic orbits on the disk around the center on the center manifold become saddle periodic orbits in the whole phase space. In fact, the former symplectic annulus (the extension of the center manifold) stays to be a symplectic annulus but now it contains a saddle with two homoclinic orbits (figure "eight"), one is small enclosing p and another is big in a neighborhood of former Γ. If Γ was orientable loop at $\varepsilon = 0$, then the saddle appeared as $\varepsilon > 0$ will asquire four orientable loops (one is small and three big ones around Γ, γ_1, γ_2). If Γ was non-orientable loop at $\varepsilon = 0$, then the saddle appeared as $\varepsilon > 0$ will have three orientable loops (around γ_1, γ_2 and one is small) and one non-orientable loop around Γ. There are also two homoclinic loops for the saddle-center appeared. They are elliptic 1-dim action orbits in the whole phase space and they will be surrounded by two dimensional tori.

What we need to add to the bifurcation pictures in order to describe the extended (saturated) neighborhoods of periodic orbits of 3 d.o.f. integrable system. It is, of course, a monodromies that can exist when go around the periodic orbit. This allows one to catch main peculiarities of the semi-global picture. One more thing that should be taken into account is resonances for elliptic singular periodic orbits which influence on the local picture when passing through this orbit. The monodromies can be taken into account when studying families of invariant tori.

Landau-Lifshitz equation is a phenomenological model describing the dynamics of magnetic media in the approximation of continual model. It was derived by L.Landau and E.Lifshitz in 1935. This equation is one of the basic models of the theory of magnetic media.



Lev Landau (1908-1968)



Eugene Lifshitz (1915-1985)

If we restrict ourself by the spatially 1-dimensional case ($x \in \mathbb{R}$, plane magnetic waves), it is written in the form

$$S_t = S \times S_{xx} + S \times JS,$$

here unitary 3-dim vector $S(x,t) = (S_1, S_2, S_3)(x,t)$, $||S|| = 1$, $J = diag(J_1, J_2, J_3)$, $J_1 < J_2 < J_3$, describes nonlinear spin waves in a ferromagnetics which move perpendicular to the anisotropy axis. This equation is integrable by the inverse scattering method (Sklyanin). Let us consider the system that describes the traveling wave solutions of the LL equation. This means the solutions of the type $S(x,t) = m(x - ut)$, where u is a velocity of the traveling wave. The the equations for the unitary vector $m(\xi)$ takes the form

$$-um' = u \times u'' + m \times Jm.$$

If one introduces new coordinates $M = um + m \times m'$, then the system is transformed as follows

$$M' + m \times Jm = 0, \quad m' = M \times m, \quad m^2 = 1, \quad M \cdot m = u. \tag{7}$$

Let us noting this system is equivalent to the well known in mechanics system which describes the motion of a solid body around its center of mass in a linear force field. At this case u has a sense of the area constant. System (7) is well studied, in particular, their integrability in the Prym theta functions was proved (Veselov, Bobenko). In papers by Pogosyan and Kharlamov, and others bifurcation diagrams were constructed and the topology of joint level of two integrals was studied. System (7) is Hamiltonian, the related symplectic structure is given by the Poisson bracket in the cotangent bundle $T^*\mathbb{R}^3 \approx \mathbb{R}^6 = \{(M,m)\}$ which is defined by the skew-symmetric matrix

$$A = \begin{pmatrix} L(M) & L(m) \\ L(m) & 0 \end{pmatrix}, \quad L(x) = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$

i.e. $\{F,G\} = (A\nabla F,\ \nabla G)$. This bracket is degenerate since two Casimir functions exist: $K = M \cdot m$ and $S = m^2$, i.e. $\{K,F\} = \{S,F\} \equiv 0$ for any smooth function $F(M,m)$).

Non degenerate symplectic structure generates by this bracket on smooth 4-dimensional submanifolds $N \subset T^*\mathbb{R}^3$ defined by equations $K = u,\ S = 1$. It is evident that N for any u is diffeomorphic to cotangent bundle T^*S^2. Hamiltonian of (7) is the function $H = (M^2 + m \cdot Jm)/2$. System (7) has an additional Clebsch integral

$$Q = (M \cdot JM - J_1 J_2 J_3 \cdot (m \cdot J^{-1} m))/2,$$

and these two integrals are independent on open dense subset in N. Thus, system (7) is Liouville integrable.

It is worth noting the additional features of the system (7). It is reversible w.r.t. involution $\sigma : (M, m) \to -(M, m)$, that is $X_H(\sigma x) = -X_H(x),\, x = (M, m)$. At $u = 0$ there is one more reversible involution $\tau : (M, m) \to (-M, m)$. The set if fixed points $Fix(\sigma)$ in N is empty but $Fix(\tau)$ coincides with the sphere $\{(0, m)\}$. Parameter u can be taken positive without a loss of generality.

Levels V_h of Hamiltonian are described by the assertion.
Proposition.

- $V_h = \emptyset$ as $h < h_c = (u^2 + J_1)/2, \quad V_h = \{C^+, C^-\}$, where $C^\pm = \pm(u, 0, 0;\ 1, 0, 0)$ at $h = h_c$;
- V_h is diffeomorphic to a disjoint union of two S^3 as $h_c < h < h_R = (u^2 + J_2)/2$, at $h = h_R$ two spheres touch each other at two points $R^\pm = \pm(0, u, 0;\ 0, 1, 0)$;,
- V_h is diffeomorphic to $S^2 \times S^1$ as $h^R < h < h_p = (u^2 + J_3)/2$, at $h = h_p$ in V_h there are two self-tangency points $P^\pm = \pm(0, 0, u;\ 0, 0, 1)$;
- V_h is diffeomorphic to $T_1^* S^2 = \mathbf{RP^3}$ as $h > h_p$, where $T_1^* S^2$ is the bundle of unit cotangent vectors, $\mathbf{RP^3}$ is the real 3-dimensional projective space.

Singular points of X_H are classified. Due to reversibility of the systems they consist of three pairs of symmetric points $C^{\pm}, R^{\pm}, P^{\pm}$,.
Eigenvalues at these points are easily calculated:
1) $C^{\pm}$: $\pm i\omega_1, \pm i\omega_2, |\omega_1| \neq |\omega_2|, \omega_i \in \mathbb{R}$ (elliptic point);
2) $R^{\pm}$: $\pm\omega, \pm\lambda, \omega, \lambda \in \mathbb{R}$, $\lambda\omega \neq 0$ (saddle-center);
3) $P^{\pm}$: $\pm\lambda_1, \pm\lambda_2, \lambda_i \neq 0;\ \lambda_1, \lambda_2 \in \mathbb{R}_1, |\lambda_1| \neq |\lambda_1|$ (saddle) as $0 \leq u < u_-,\ \pm(\alpha \pm i\beta),\ \alpha\beta \neq 0, i = \sqrt{-1}, \alpha, \beta \in \mathbb{R}$ as $u_- < u < u_+$ (focus-focus), $\pm i\nu_1, ; \pm i\nu_2,\ \nu_1 \cdot \nu_2 \neq 0,\ \nu_1, \nu_2 \in \mathbb{R},\ |\nu_1| \neq |\nu_2|$ as $u > u_+$, here $u_{\pm} = \sqrt{J_3 - J_1} \pm \sqrt{J_3 - J_2}$.
For points saddle-saddle and focus-focus there are two local two-dimensional stable and unstable manifolds Continuation of these manifolds gives global stable and unstable manifolds. For saddle-center points these manifolds also exist but they are one-dimensional.

From the physical applications of equation (7), important role play orbits which belong to the intersection of stable and unstable manifolds of different or the same singular points, that is hetero- or homoclinic orbits. For the initial LL system they correspond to the traveling waves of a soliton type. Physically, they are domain walls, that is sharp boundary between domains of the different magnetization, so-called domains. In the theory of LL equation such solutions are called topological (for heteroclinic orbits, going to the different symmetric equilibria) or nontopological (for homoclinic orbits) solitons.
When the magnetic energy is perturbed by the form of the fourth order $\nu(m \cdot J_0 m) m \times J_0 m,\ J_0 = \mathrm{diag}\,(-1,-1,0)$ in the first equation of (7), separatrix surfaces are split and complicated set of solitons appear. Here to find splitting a formula found by us (L.-Umanskiy) is applicable.

As an example there all described exist is the well known in mechanics system of the movement of the Kowalewski top in the two forces field (Bogoyavlensky, Kharlamov, Reiman-Semenov-Tian-Shan'sky)

$$\dot{\omega}_1 = (\omega_1\omega_2 + \beta_3)/2,\ \dot{\alpha}_1 = \alpha_2\omega_3 - \alpha_3\omega_2,\ \dot{\beta}_1 = \beta_2\omega_3 - \beta_3\omega_2,$$
$$\dot{\omega}_2 = (-\omega_1\omega_3 - \alpha_3)/2,\ \dot{\alpha}_2 = \alpha_3\omega_1 - \alpha_1\omega_3,\ \dot{\beta}_2 = \beta_3\omega_1 - \beta_1\omega_3,$$
$$\dot{\omega}_3 = \alpha_2 - \beta_1,\ \dot{\alpha}_3 = \alpha_1\omega_2 - \alpha_2\omega_1,\ \dot{\beta}_3 = \beta_1\omega_2 - \beta_2\omega_1,$$

here ω is the vector of instant angular velocity, vectors α, β characterize field forces, $||\alpha||^2 = a^2$, $||\beta||^2 = b^2$, $\alpha \cdot \beta = 0$

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