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The Clebsch Top

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(joint work with Taras Skrypnik)

1. The Clebsch equations.

The topic of this talk is a old problem of classical mechanics, related with movement of a rigid body with a fixed point immersed in an inviscid fluid, suggested by Clebsch in 1870. It is therefore called the Clebsch top.

According to him, the behaviour of

such a body is described by a system of six first-order ODE's. By a suitable choice of coordinates these equations are

$$\dot{S}_1 = (j_3 - j_2) T_3 T_2$$

$$\dot{S}_2 = (j_1 - j_3) T_1 T_3$$

$$\dot{S}_3 = (j_2 - j_1) T_2 T_1 \qquad (I)$$

$$\dot{T}_1 = T_3 S_2 - T_2 S_3$$

$$\dot{T}_2 = T_1 S_3 - T_3 S_1$$

$$\dot{T}_3 = T_2 S_1 - T_1 S_2$$

They depend on three arbitrary parameters $j_1 < j_2 < j_3$, and have a quadratic nonlinearity. They are difficult to solve. The purpose of my talk is to show a relatively simple way of solving these equations, inspired by some ideas of bihamiltonian geometry.

Before discussing the solution, let me recall some properties of the Clebsch equations. They concern :

1. the integrals of motion
2. the Hamiltonian structure
3. the Lax representation

First, one may notice that the Clebsch eq.ns admit four quadratic integrals of motion :

$$C_1 = S_1 T_1 + S_2 T_2 + S_3 T_3$$

$$C_2 = T_1^2 + T_2^2 + T_3^2$$

$$H = S_1^2 + S_2^2 + S_3^2 + (j_2 + j_3) T_1^2 + (j_3 + j_1) T_2^2 + (j_1 + j_2) T_3^2$$

$$K = j_1 S_1^2 + j_2 S_2^2 + j_3 S_3^2 + j_2 j_3 T_1^2 + j_3 j_1 T_2^2 + j_1 j_2 T_3^2 .$$

Secondly, one may notice that these equations are also Hamiltonian. Indeed (and not surprisingly) the phase space of the Clebsch top may be identified with $e(3)^*$, endowed with the canonical Poisson bracket

$$\{S_\alpha, S_\beta\} = \varepsilon_{\alpha\beta\gamma} S_\gamma \quad \{S_\alpha, T_\beta\} = \varepsilon_{\alpha\beta\gamma} T_\gamma \quad \{T_\alpha, T_\beta\} = 0 .$$

The Hamiltonian function is the integral H. One may also notice that C_1 and C_2 are Casimir functions of this Poisson bracket, and that H and K commute. The conclusion is that the Clebsch top is an integrable Hamilto.

nian system with two degrees of freedom on $e(3)^*$.

Another remark is that the Clebsch eq.ns admit a Lax representation. The Lax matrix is

$$L(u) = \sum_{\alpha=1}^{3} L_\alpha(u) \cdot \sigma_\alpha$$

where σ_α are the Pauli matrices, and the coefficients $L_\alpha(u)$ (depending on the spectral parameter u) are given by

$$L_\alpha(u) = u_\alpha S_\alpha + u_\beta u_\gamma T_\alpha \, .$$

Here appear for the first-time the irrational quantities

$$u_\alpha = \sqrt{u + j_\alpha}$$

which plague the theory of Clebsch eq.ns. The associate spectral curve is

$$\Gamma_1 : \quad \mu^2 + (C_2 u^2 + Hu + K) + 2C_1 (u_1 u_2 u_3) = 0$$

It contains all the integrals of the motion, packed in an unexpected way due to the presence of the last irrational coefficient.

Another remarkable property of the Clebsch eq.ns is that they are bihamiltonian and possess a second Lax matrix. In a sense, everything is doubled. The second Poisson bracket depends explicitly on the arbitrary parameters j_α

according to:

$$\{S_\alpha, S_\beta\}_2 = \varepsilon_{\alpha\beta\gamma}\, j_\gamma S_\gamma \quad \{S_\alpha, T_\beta\} = \varepsilon_{\alpha\beta\gamma}\, j_\beta T_\gamma \quad \{T_\alpha, T_\beta\} = \varepsilon_{\alpha\beta\gamma} S_\gamma$$

while the second Lax matrix (which I don't write here explicitly) leads to a second spectral curve of degree four:

$$\Gamma_2 : \quad \mu^4 + \mu^2 (C_2 u^2 + Hu + K) + C_1^2 (u+j_1)(u+j_2)(u+j_3) = 0 .$$

A last remark is appropriate concerning the bihamiltonian formulation of the Clebsch eq.ns. If one consider the Poisson pencil $Q + uP$ (where Q and P are the second and the first Poisson bivector respectively), one may notice that it is a Poisson pencil of Kronecker type (in the terminology of Gelfand-Zakharevich). It admits two distinct polynomial Casimir function

$$C_1(u) = C_1 \qquad \text{order } 0$$

$$C_2(u) = C_2 u^2 + k u + k \qquad \text{order } 2$$

So, the Casimir functions are kept distinct by the Poisson pencil, while they are packed in some strange way by the spectral curves.

About the history of the Clebsch eq.ns it should be appropriate to mention the names of Minkowski, Schottky, Kotter, Adler-van Moerbeke, Sklyanin-Takebe, but I will omit any historical comment for brevity.

2. How to solve the Clebsch equations.

Let $v: e(3)^* \to \mathbb{R}$ be the function implicitly defined (in a suitable open domain) by the equation

$$\sum_{\alpha=1}^{3} c_\alpha (v_\alpha S_\alpha + v_\beta v_\gamma T_\alpha) = 0 \qquad (\mathrm{II})$$

where

$$v_\alpha = \sqrt{v + \dot{j}_\alpha}$$

as usual, and

$$c_\alpha^2 = \dot{j}_\beta - \dot{j}_\gamma$$

are the constants appearing in the equation of motion. One may notice the close connection of the basic Eq. II with the first Lax matrix. Indeed this equation may be written in the form

$$\sum_{\alpha=1}^{3} c_\alpha L_\alpha(V) = 0 . \qquad (\text{II bis})$$

It means that the value $v_\alpha(S,T)$ of the function v, at each point (S_α, T_α) of the phase space, is the value of the spectral parameter of the Lax matrix $L(u)$ which annihilates the weighted linear combination $\sum_\alpha c_\alpha L_\alpha(u)$ of its entries.

With the function v defined by the

above equation, let us construct the allied function

$$E = \sum_{\alpha} c_\alpha v_\alpha T_\alpha$$

$$F = \sum_{\alpha} c_\alpha \left[v_\beta v_\gamma S_\alpha + (j_\beta + j_\gamma) v_\alpha T_\alpha \right] \qquad (\mathrm{III})$$

$$G = \sum_{\alpha} c_\alpha \left[v_\beta v_\gamma j_\alpha S_\alpha + j_\beta j_\gamma v_\alpha T_\alpha \right]$$

They may be used to construct the second-order polynomial

$$P(x) = E x^2 + F x + G \quad . \qquad (\mathrm{IV})$$

The roots of this polynomial are

$$x_1 = +V \qquad (\mathrm{V})$$

$$X_2 = -V + \frac{F}{E} \quad ,$$

These roots allow to solve the Clebsch equation.
I will show three remarkable properties.

1. Abel form :

Let us choose $(x_1, x_2 ; C_1, C_2, H, K)$ as a system of coordinates adapted to the Lagrangean foliation. In these coordinates the Clebsch eq.ns assume the Abel form

$$\frac{\dot{x}_1}{R_1(x_1,y_1)} + \frac{\dot{x}_2}{R_2(x_2,y_2)} = 0$$

$$\frac{x_1\dot{x}_1}{R_1(x_1,y_1)} + \frac{x_2\dot{x}_2}{R_2(x_2,y_2)} = 1$$

where the rational functions

$$R_1(x,y) = \frac{1}{8y\,(x+j_1)(x+j_2)(x+j_3)}$$

$$R_2(x,y) = 2y\,(x+j_1)(x+j_2)(x+j_3) - 2\,\frac{C_1^2}{y^3}$$

must be evaluated on the algebraic curve

$$\Gamma_1 : \quad \left[4(x+j_1)(x+j_2)(x+j_3)y^2 + (C_2x^2+Hx+K) \right]^2 = 4C_1^2(x+j_1)(x+j_2)(x+j_3)$$

$$\Gamma_2 : \quad (x+j_1)(x+j_2)(x+j_3)y^4 + 4(C_2x^2+Hx+K)y^2 + C_1^2 = 0$$

respectively.

The noticeable feature of this result is the appearance of two algebraic curves in the Abel form of the equations of motion, contrary to the standard canon of the Lax theory.

where only one algebraic curve is usually admitted.

2. Jacobi form.

In the same coordinates it is possible also to solve the HJ equation of the Clebsch top. For that one must notice that the roots x_1 and x_2 commute with respect to both Poisson brackets, and that the conjugate momenta are provided by the equations:

$$4(x_1+j_1)(x_1+j_2)(x_1+j_3)p_1^2 + (C_2x_1^2+Hx_1+K) + 2C_1\sqrt{(x_1+j_1)(x_1+j_2)(x_1+j_3)} = 0$$

$$2(x_2+j_1)(x_2+j_2)(x_2+j_3)p_2^2 + (C_2x_2^2+Hx_2+K) + \sqrt{(C_2x_2^2+Hx_2+K)^2 - 4(x_2+j_1)(x_2+j_2)(x_2+j_3)C_1^2} = 0$$

Therefore, by solving the pair of ODE's

$$4(x_1+j_1)(x_1+j_2)(x_1+j_3)\left(\frac{dW_1}{dx_1}\right)^2 + (C_2x_1^2+Hx_1+K)+2C_1\sqrt{(x_1+j_1)(x_1+j_2)(x_1+j_3)}=0$$

$$2(x_2+j_1)(x_2+j_2)(x_2+j_3)\left(\frac{dW_2}{dx_2}\right)^2+(C_2x_2^2+Hx_2+K)+$$
$$+\sqrt{(C_2x_2^2+Hx_2+K)^2-4C_1^2(x_2+j_1)(x_2+j_2)(x_2+j_3)}=0$$

and by setting

$$W(x_1,x_2;a_1,a_2) = W_1(x_1,a_1) + W_2(x_2,a_2)$$

one obtains a complete integral of the stationary HJ equation of the Clebsch top.

8. Reconstruction formulas

It is also possible to reconstruct explicitly the time evolution of the mechanical coordinates (S_α, T_α). Indeed the following reconstruction formulas hold true:

$$S_\alpha = -\frac{c_\alpha}{c_1^2 c_2^2 c_3^2}\left[\left((j_1+j_2+j_3)g_1+g_2\right)v_\alpha + g_3 v_\beta v_\gamma - g_1 v_\alpha^3\right]$$

$$T_\alpha = -\frac{c_\alpha}{c_1^2 c_2^2 c_3^2}\left[\left((j_1+j_2+j_3)f_1+f_2\right)v_\alpha + f_3 v_\beta v_\gamma - f_1 v_\alpha^3\right]$$

where the coefficients $(f_1, f_2, f_3; g_1, g_2, g_3)$ are given by:

$$f_1 = c_1 c_2 c_3 p_2$$

$$-2 f_1 f_2 = c_1^2 c_2^2 c_3^2 C_2 - f_1^2 (x_1 + j_1 + j_2 + j_3) + f_3^2$$

$$(x_1 - x_2) f_3 = c_1 c_2 c_3 \left[- \frac{c_1}{p_2} + (2p_1 - p_2) \sqrt{(x_1 + j_1)(x_1 + j_2)(x_1 + j_3)} \right]$$

$$g_1 = - f_3$$

$$g_2 = 2 c_1 c_2 c_3 (x_1 + j_1)(x_1 + j_2)(x_1 + j_3) + x_1 f_3 - f_1 \sqrt{(x_1 + j_1)(x_1 + j_2)(x_1 + j_3)}$$

$$g_3 = -(x_1+x_2+y_1+y_2+y_3)f_1 + f_2 .$$

$\square$

3. Where is this result coming from?

As I said before, this way of solving the Clebsch equation has been suggested by certain ideas of bihamiltonian geometry. The main steps of the procedure are summarized in the following synoptic table (which is not explicative of the origin of the method):

1. The equations
2. The Poisson structure
3. The Poisson cohomology
4. The 1-cocycle
5. The compatibility of the 1-cocycle with the Hamiltonian ("Kow conditions")

6. The polynomial $P(x)$

7. The conjugate moments

8. The Abel equations

9. The reconstruction formulas.

4. How to check this result.

Even if the strategy is reasonably clear and well-defined, to perform explicitly all the computations is a feasible task. This is due to the fact that the "generating function" v is only implicitly defined, and that a lot of radicals enter in any equations. There are two strategies to acquaint for this problem.