The moduli space of semitoric integrable systems

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Toric and semitoric integrable systems

An *integrable system* is a triple $(M, \omega, F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n)$ such that (M, ω) is a $2n$-dimensional symplectic manifold and $(f_i : M \to \mathbb{R})_{i=1}^n$ is a set of functions on M which are independent almost everywhere and which Poisson commute.

A *toric integrable system* (or *toric manifold*) is an integrable system such that each f_i is a momentum map for a Hamiltonian S^1-action with fixed period. It is a classical result that such systems are classified by the image of F, which is a convex polygon in $\mathbb{R}^n$ (proven in the 1980s by Atiyah, Guillemin-Sternberg, and Delzant).

In [3, 4] Pelayo-Vũ Ngọc generalize this idea. A *semitoric integrable system* (or *semitoric manifold*) is a 4-dimensional integrable system $(M, \omega, F = (J, H))$ such that only J is required to be a proper momentum map for a Hamiltonian S^1-action on M, plus a few other technical requirements.

The classification of semitoric systems

Semitoric systems are classified by five invariants. A *list of semitoric ingredients* [3, 4] is the following:

1. *The affine invariant*: An infinite family of polygons;
2. *The number of singular points invariant*: The number of focus-focus points;
3. *The Taylor series invariant*: A Taylor series for each focus-focus point;
4. *The volume invariant*: A real number encoding the location of each singular fiber;
5. *The twisting index*: An integer assigned to each pair of focus-focus points.

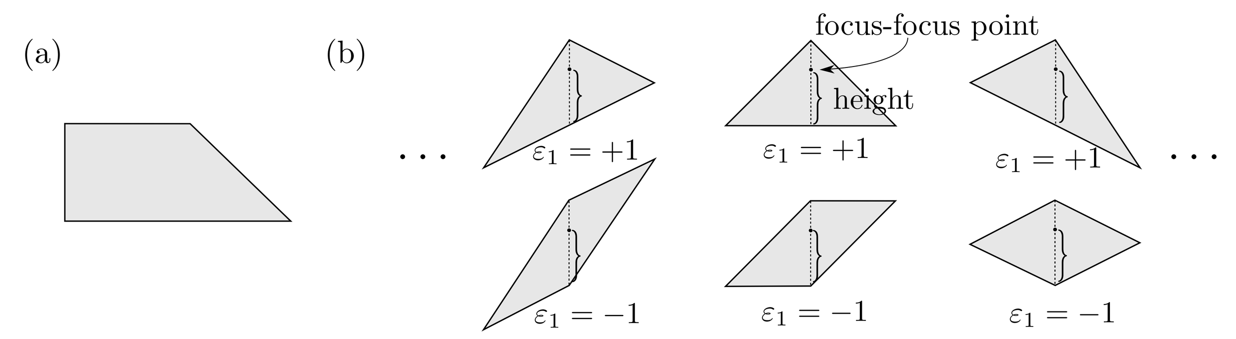

Figure 1: (a) The complete invariant for a toric system is a single convex polygon. (b) The complete invariant for a semitoric system is an infinite family of polygons labeled with extra information. The Taylor series invariant is not pictured here and the volume invariant is equal to the height of the focus-focus point from the bottom of the polygon.

The complete invariant can be thought of as an infinite family of polygons which have distinguished interior points each labeled with a Taylor series, a real number (the volume invariant), and an integer (related to the twisting index). These points correspond to the focus-focus singular points of the system.

The polygons are naturally organized into a countable collection of finite families based on the choice of initial action-angle coordinates. In each family there is one preferred element known as a *primitive semitoric polygon*.

The Taylor series completely determines the structure of the focus-focus point in a neighborhood of the singular fiber and is constructed by following the Hamiltonian flows of J and H on fibers close to the singular fiber. The twisting index encodes the twistedness of the singular Lagrangian fibration between two focus-focus points.

Classification Theorem for Semitoric Systems ([3, 4]). *There exists a bijection*

$$\Phi : \mathcal{M}_{\mathrm{ST}} \to \mathcal{I}$$

where $\mathcal{M}_{\mathrm{ST}}$ is the moduli space of simple semitoric systems and $\mathcal{I}$ is the collection of semitoric ingredients.

This bijection is made precise in [3, 4].

Results

Let $\mathcal{M}_{\mathrm{ST}}$ denote the moduli space of simple semitoric systems. In [2] a metric $\mathcal{D}$ is defined on this space. This metric depends on two parameters (a nonstandard measure on $\mathbb{R}^2$ and a sequence of real numbers).

Theorem 1 ([2]). *For any admissible choice of parameters the space $(\mathcal{M}_{\mathrm{ST}}, \mathcal{D})$ is a non-complete metric space and the topology induced by $\mathcal{D}$ on $\mathcal{M}_{\mathrm{ST}}$ does not depend on the choice of parameters.*

In [1] we use algebraic techniques to recover known results about toric systems and prove the following about semitoric systems.

Theorem 2 ([1]). *For any choice of number of singularities $m_f \in \mathbb{Z}_{\geq 0}$ and twisting index $\vec{k} \in \mathbb{Z}^{m_f}$ the space of semitoric systems with that twisting index and number of singularities is connected.*

Metric space structure

In [2] a metric $\mathcal{D}$ is defined on $\mathcal{M}_{\mathrm{ST}}$ by pulling back a metric defined on $\mathcal{I}$ via the map Φ from the classification theorem.

Roughly, two lists of ingredients are compared in the following way:

1. if the number of focus-focus points and twisting index are not compatible then the two systems are defined to be in different components of $\mathcal{M}_{\mathrm{ST}}$;
2. the Taylor series invariants are compared term by term, with higher order terms having lower weight;
3. the distance between the heights of corresponding distinguished points in the polygons is computed;
4. the families of polygons are compared by taking the volume of the symmetric difference of a specific finite subset of the polygons.

Toric and semitoric fans

By *toric fan* we mean a finite collection of vectors in $\mathbb{Z}^2$ which can be realized as the inwards pointing primitive normal vectors to a 2-dimensional Delzant polygon (this is a special case of a fan associated to a toric variety).

We define a *semitoric fan* to be a collection of primitive inwards pointing normal vectors to a primitive semitoric polygon.

To study the moduli space we will depend on transformations of the fan which can be realized as continuous families of systems, such as the *corner chop* operation.

The corner condition on Delzant polygons forces any toric fan to satisfy a collection of linear equations parameterized by some $a_0, \ldots, a_{d-1} \in \mathbb{Z}$. These integers must satisfy

$$\begin{pmatrix} 0 & -1 \\ 1 & a_0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & a_{d-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (1)$$

but there are collections of vectors which are not toric fans but whose associated integers satisfy Equation (1).

In fact, there are many sequences of integers which satisfy Equation (1) but do not correspond to a toric fan, so such lists of integers cannot be used to classify toric fans. In short, the problem is that Equation (1) does not account for the number of times that the vectors wind around the origin.

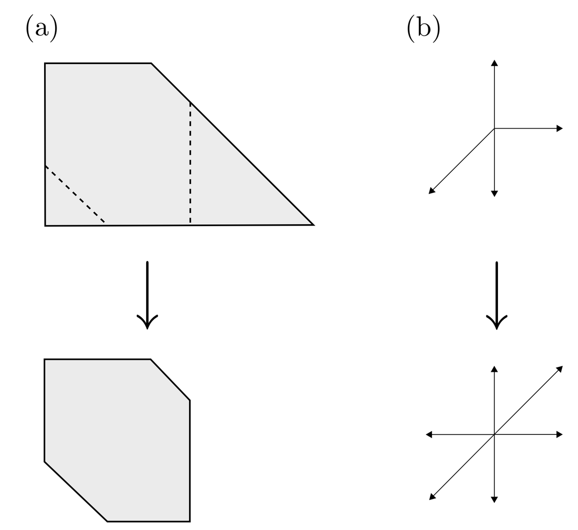

Figure 2: The corner chop operation on (a) a polygon and (b) the associated fan.

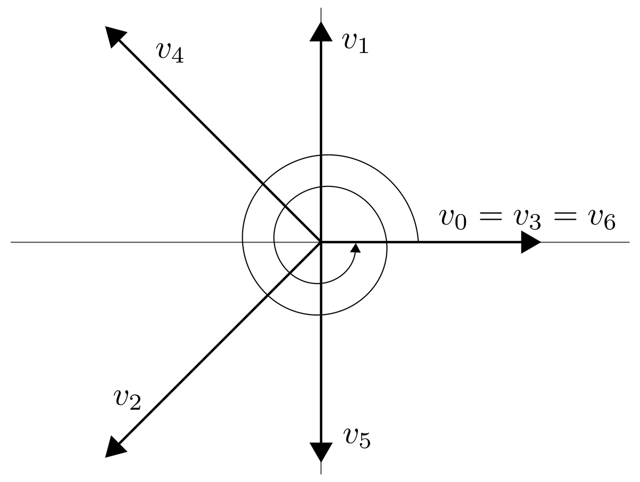

Figure 3: This is not a toric fan but the associated integers do satisfy Equation (1).

Lifting Equation (1) from $\mathrm{SL}_2(\mathbb{Z})$

Let G denote the preimage of $\mathrm{SL}_2(\mathbb{Z})$ in the universal cover of $\mathrm{SL}_2(\mathbb{R})$. Then we have

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \mid STS = T^{-1}ST^{-1},\ S^4 = I \rangle \text{ and}$$
$$G \cong \langle S, T \mid STS = T^{-1}ST^{-1} \rangle$$

for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and Equation (1) becomes $ST^{a_0} \cdots ST^{a_{d-1}} = I$.

The group G acts as the universal cover of $\mathrm{SL}_2(\mathbb{Z})$ and thus is sensitive to the *winding number* of an element.

The group relations are shown in Figure 4 where $\mathbb{Z} * \mathbb{Z}$ denotes the free group on S and T, each π_i is a projection, and the map $w : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$ takes an element to 12 times its winding number.

Now we lift Equation (1) to

$$ST^{a_0} \cdots ST^{a_{d-1}} = S^4 \text{ (in } G). \quad (2)$$

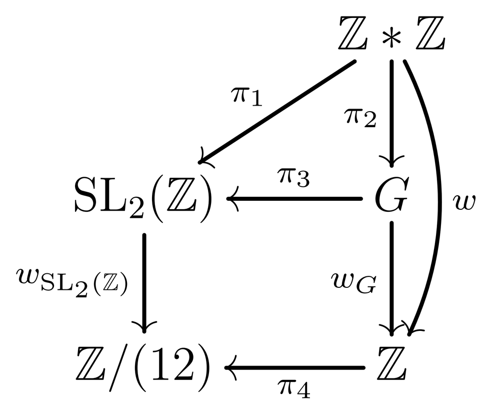

Figure 4: G is the fiber product of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{Z}$ over $\mathbb{Z}/(12)$.

Proposition 1 ([1]). *There is a one-to-one correspondence between lists of integers which satisfy Equation (2) in G and toric fans (up to isomorphism).*

Similar, but more delicate, results hold for the semitoric case.

The connected components of $\mathcal{M}_{\mathrm{ST}}$

The strategy to prove Theorem 2 is as follows:

1. associated to any semitoric system is a semitoric fan;
2. associated to the semitoric fan is a list of integers $a_0, \ldots, a_{d-1} \in \mathbb{Z}$ which define an element $\sigma = ST^{a_0} \cdots ST^{a_{d-1}} = S^4$ in G;
3. reduce σ to a standard form by using only reductions of a specific type which correspond to continuous transitions of the polygon (such as a corner chop);
4. conclude that any system which can be reduced to the same standard form in this way must be in the same connected component of $\mathcal{M}_{\mathrm{ST}}$.

Acknowledgements

The author is grateful to his advisor Álvaro Pelayo for guidance during these projects and for originally suggesting the questions studied in [1, 2]. The author and his collaborator Á. Pelayo were supported by the grants DMS-1055897 and DMS-1518420 while working on some of the results presented on this poster.

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