Almost-toric systems: description and classification

Christophe Wacheux, christophe.wacheux@epfl.ch

École Polytechnique Fédérale de Lausanne

Abstract

We give here the existing results concerning the classification & description of integrable Hamiltonian systems for which certain component yield periodic flows (and with restrictions on critical points).

First definitions

Definition: integrable Hamiltonian systems

An integrable Hamiltonian system (or IHS) is a symplectic manifold (M^{2n},ω) and a family of n functions $F=(f_1,\dots,f_n)\in\mathcal{C}^\infty(M^{2n}\to\mathbb{R}^n)$, such that:

- $\forall 1\leqslant i,j\leqslant n,\ \{f_i,f_j\}=0$,
- The rank of dF is maximal on a dense set

Motivation and first results

Theorem : Atiyah – Guillemin & Sternberg,1982

Let $\mathbb{T}^d \circlearrowright M^{2n}$ be a (possibly not effective) Hamiltonian torus action, with moment map $F: M\to\mathbb{R}^d$. We have that:

- the fibers of F are connected,
- $F(M)$ is a rational convex polytope called *moment polytope*. It is the convex hull of the image by F of the fixed points of the $\mathbb{T}^d$-action.

Theorem (Delzant, 1988, [Del88],[Del90])

For $d=n$, if the action is effective, the moment polytope is normal: the edges of the polytopes originating at each vertex form a basis of $\mathbb{R}^n$. It characterizes completely the toric system (M,ω,F) up to $\mathbb{T}^n$-equivariant symplectomorphism.

Bifurcation diagram: with $F(M)$, one can detect the rank of F at a given $p\in M$ and hence the dimension of its orbit. If $F(p)$ is on a k-facet, the orbit is a k-torus.

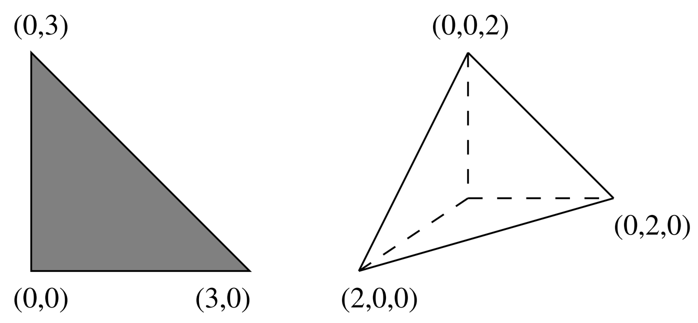


Fig. 1: Standard Hamiltonian action of $\mathbb{T}^2$ on $\mathbb{CP}^2$

We have a classification by "simple" objects: the *Delzant* polytopes.

Can we extend this classification to integrable Hamiltonian systems for which one (or more) component(s) is non-peridic ?

Definition of almost-toric IHS

A fixed point $p\in M$ for F, is called non-degenerate if the Lie algebra spawned by the Hessians of the f_i's is a Cartan subalgebra of $\mathfrak{sp}(2n)$. A critical point is non-degenerate if, once taken the symplectic quotient by the torus action of the regular components of F, its image is a fixed point of the resulting IHS.

Theorem (Williamson, 1936 [Wil36])

Let $p\in M$ a nondegenerate critical point in of F an IHS. Then there exists a quadruplet $\Bbbk=(k_\mathrm{e},k_\mathrm{f},k_\mathrm{h},k_\mathrm{x})\in\mathbb{N}^4$, an open set $\mathcal{U}_p$ of p and a symplectomorphism $\varphi:(\mathcal{U}_p,\omega,p)\to(\mathbb{R}^{2n},\omega_0=\sum_{i=1}^n d\xi_i\wedge dx_i,0)$ such that

$$\varphi^*F=Q_\Bbbk+o(2),\text{ with }Q_\Bbbk=(\mathbf{e}_{k_\mathrm{e}},\mathbf{h}_{k_\mathrm{h}},\mathbf{f}_{k_\mathrm{f}},\xi_{n-k_\mathrm{x}+1},\dots,\xi_n)\text{ and}$$

- $\mathrm{e}_i=x_i^2+\xi_i^2$ - elliptic (or E) components,
- $\mathrm{h}_i=x_iy_i$ - hyperbolic (or H) components,
- $f_i=(f_i^1,f_i^2)$, $\begin{cases}\mathrm{f}_i^1=x_i^1\xi_i^1+x_i^2\xi_i^2\\ \mathrm{f}_i^2=x_i^1\xi_i^2-x_i^2\xi_i^1\end{cases}$ - focus-focus (or FF) components.

We define $(\mathcal{W}(F),\preccurlyeq)$ as the (partially) ordered set of Williamson types $\Bbbk$ that occur in a given IHS. Note that k_x is actually the rank of F at p.

Definition: almost-toric IHS

An IHS (M,ω,F) is called **almost-toric** of complexity $0\leqslant c\leqslant n$ if:

- all critical points are non-degenerate,
- there are no singularities of hyperbolic type: $k_\mathrm{h}=0$,
- the joint flow of $\check{F}^c:=(f_{c+1},\dots,f_n)$ is 2π-periodic: it generates a Hamiltonian $\mathbb{T}^{n-c}$-action.

If $c=0$, the system is called *toric*, and *semi-toric* if $c=1$. In this case we note $\check{F}=\check{F}^1$

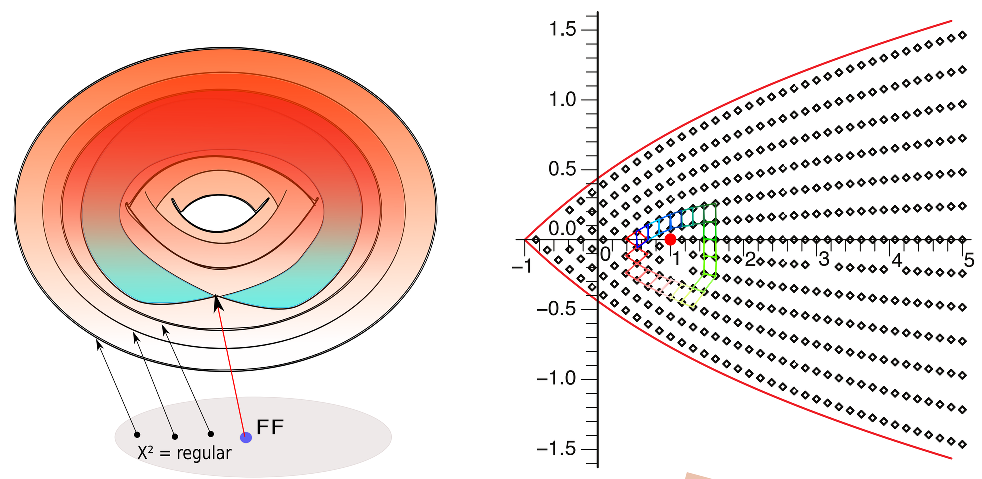
1. **What are the general features of $F(M)$? Is it still a convex polytope ? What can we know about the dynamics of the IHS at a point p from its image $F(p)$?**
2. **Does $F(M)$ classify entirely the almost-toric systems (up to some fitting equivalence relation) ?**

Problems with almost-toric systems

From Action-Angle theorem, we know that on regular leaves, $\mathcal{F}$ is isomorphic to a foliation by Lagrangian tori and $F(M)$ has a natural integral affine structure. At focus-focus singularities, $\mathcal{F}$ pinches (one or several times), and the integral affine structure has non-trivial monodromy (right).

Atiyah - Guillemin & Sternberg only apply partially, and Delzant theorems do not apply anymore *a priori*. What is left of it ? i.e.: can we try to recover a classification "à la Delzant" for almost-toric systems ?

Foliation and integral affine structure near a focus-focus singularity ($2n=4$ and $c=1$).



From now on, all the results are only for semi-toric systems, with a single pinch.

Recovering moment polytopes

In the toric case, the integral affine structure of $F(M)$ is locally convex and possess a global section, which thus "straightens" $F(M)$ to a Delzant polytope. A way to straighten $F(M)$, is to trivialize the monodromy by "cutting" the image, thus making it simply connected. The global S^1-action, gives us a privileged direction to ensure that the resulting polytope is Delzant.

Theorem (San Vũ Ngọc, 2003, [VN07])

Given a semi-toric IHS $(M^4,\omega,F=(J,H))$ with $m_\mathrm{f}\in\mathbb{N}^*$ focus-focus critical points, we have that the fibers of F are connected, and we can cut $F(M)$ on a vertical half-line at each focus-focus value such that it straightens to a Delzant polygon.

We recover a Delzant polytope, but because we have a choice for the cut (up or down), we obtain, for a given semi-toric IHS, a family of Delzant polytopes. This family is the first invariant for the classification. It is actually the orbit for a free and transitive $(\mathbb{Z}/(2\mathbb{Z}))^{m_f}$-action.

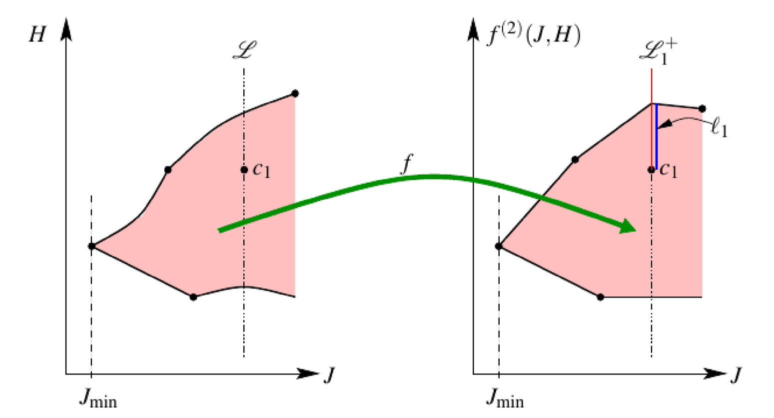


Fig. 3: Here, f is the straightening map for $F(M)$. It depends on the choice of the cuts. (credits goes to San Vũ Ngọc for the picture)

Classification & description of semi-toric systems in dimension 2n=4

Theorem (Pelayo, San Vũ Ngọc [PVN09],[PVN11])

On (M^4,ω), the semi-toric integrable systems are classified by a finite number of invariants:

- The number m_f of focus-focus points and their position,
- The family of rational convex polygon given by the cutting of $F(M)$.
- For each focus-focus critical leaf, a Taylor serie S^∞ in two variables,
- The volume invariant: obtained from $F(M)$ and the Duistermaat-Heckan measure,
- The twisting index.

All these invariants can *interpretated* as parameters of different features of the semi-toric IHS. This gives us a description of semi-toric IHS.

An Atiyah - Guillemin & Sternberg theorem for semi-toric systems

We can apply Atiyah - Guillemin & Sternberg to $(P^{(i)}_{\preccurlyeq\Bbbk}(M),\omega^{(i)}_{\preccurlyeq\Bbbk},\check{F}_\Bbbk)$: the image $\check{\Delta}^{(i)}_\Bbbk:=\check{F}(P^{(i)}_{\preccurlyeq\Bbbk}(M))$ is a $\check{k}^{(i)}_\mathrm{x}$-dimensional polytope.

Theorem (W., 2013, [Wac13], [Wac15])

We have the following alternative:

a. either $\check{k}^{(i)}_\mathrm{x}=k_\mathrm{x}$ (Transverse stratum):

$$\min_{(\check{F}^{(i)}_\Bbbk)^{-1}(\check{q})} f_1 =: m^{(i)}_\Bbbk(\check{q})=M^{(i)}_\Bbbk(\check{q}) =: \max_{(\check{F}^{(i)}_\Bbbk)^{-1}(\check{q})} f_1,$$

$$m^{(i)}_\Bbbk, M^{(i)}_\Bbbk\in\mathcal{C}^\infty(\check{\Delta}^{(i)}_\Bbbk,\mathbb{R})\text{ and }V^{(i)}_\Bbbk=graph\left(m^{(i)}_\Bbbk\right),$$

b. or $\check{k}^{(i)}_\mathrm{x}=k_\mathrm{x}+1$ (Vertical stratum):

$$V^{(i)}_\Bbbk=epigraph\left(m^{(i)}_\Bbbk\right)\cap hypograph\left(M^{(i)}_\Bbbk\right)$$

with $m^{(i)}_\Bbbk, M^{(i)}_\Bbbk$ only **piece-wise** $\mathcal{C}^\infty(\check{\Delta}^{(i)}_\Bbbk\to\mathbb{R})$.

Theorem (W., 2013-...)

The fibers of F are connected.

Our proof in any dimension relies on the stratification result. Hence the general case should be published along with it.

Stratification of semi-toric IHS

Given an almost-toric IHS We note $P_\Bbbk(M)$ the set of points of M^{2n} of a given $\Bbbk$, $V_\Bbbk(M)=F(P_\Bbbk(M))$, with an $^{(i)}$ for each connected component.

Theorem (W., 2013-...)

Let (M,ω,F) be an semi-toric IHS. We have:

1. The map $\mathbf{P}(\mathbf{F}):\Bbbk\mapsto P_\Bbbk(M)$ stratifies M by **symplectic** sub-manifolds,
2. the restriction $F^{(i)}_\Bbbk$ of F to each connected components of the skeleta $P^{(i)}_{\preccurlyeq\Bbbk}$ gives a semi-toric systems of dimension $2k_\mathrm{x}$.

We have thus a $\mathcal{W}(F)$-stratification of (M,ω,F) **by semi-toric systems**.

Item 1. is proved in [Wac15], item 2. is proven for $2n=6$ in [Wac13], the general version shall be available soon.

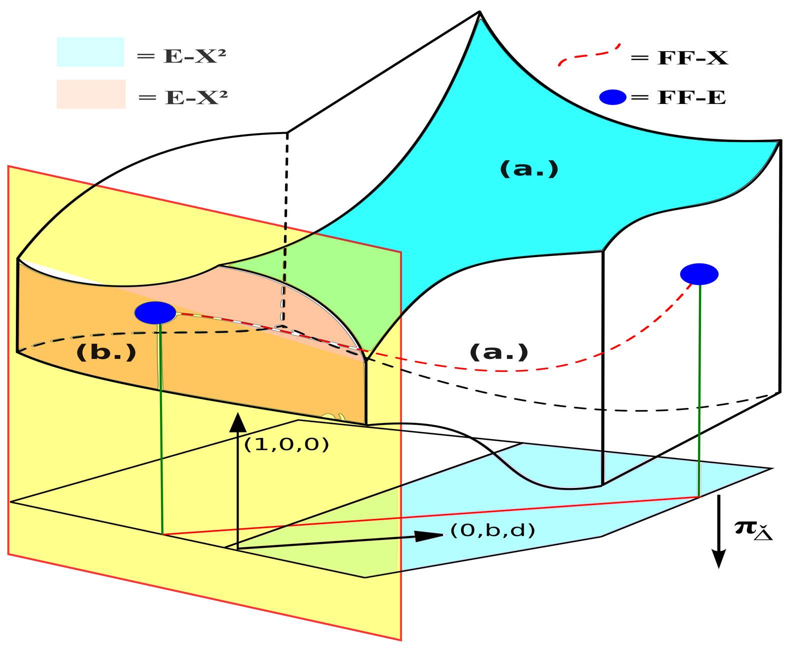


Fig. 4: Image of the moment map for a $2n=6$ semi-toric IHS. The colors gives the Williamson type of the critical value.

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