Integrability via reversibility

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**Finite Dimensional Integrable System
in Geometry and Physics, Bedlewo**

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Outline

Is there a real-analytic volume preserving dynamical system on a compact manifold with an open set foliated by invariant tori carrying quasi-periodic motions, and positive Lyapunov exponents everywhere in the interior of the complement?

The Lie group

$$G \ni \begin{bmatrix} 1 & 0 \\ w & e^{uL} \end{bmatrix}, \quad w \in \mathbb{R}^n, u \in \mathbb{R}.$$

where L is a fixed $n \times n$ matrix.

Semi-direct product of $\mathbb{R}^n$ and $\mathbb{R}$

$$(w, u) \in \mathbb{R}^n \rtimes \mathbb{R}, \qquad (w_1, u_1)(w_2, u_2) = \left(w_1 + e^{u_1 L} w_2,\ u_1 + u_2\right)$$

$\mathcal{K} : G \to G, \quad \mathcal{K}(w, u) = (-w, u)$ is an automorphism of G.

The Lie algebra

$$\mathfrak{g} \ni \begin{bmatrix} 0 & 0 \\ \xi & \eta L \end{bmatrix}, \quad \xi \in \mathbb{R}^n, \eta \in \mathbb{R}.$$

$K : \mathfrak{g} \to \mathfrak{g}, \quad K(\xi, \eta) = (-\xi, \eta)$ is an automorphism of $\mathfrak{g}$

There are very few Lie groups with this kind of automorphism.

(If $\mathcal{K} = -I$ is an automorphism then G is abelian!)

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Left invariant metric in G defined by the Euclidean metric $\xi^2 + \eta^2$ in $\mathfrak{g}$.

$$\left(e^{-uL}dw\right)^2 + du^2$$

$$TG = G \times \mathfrak{g} \ni (x; v) = (w, u; \xi, \eta)$$

Noncompact locally homogeneous space

$$M \ni (w, u) \quad w \quad mod \quad 1 \qquad (mod \quad \Gamma_0)$$

where Γ_0 is a rank n lattice in $\mathbb{R}^n$.
$M = \Gamma_0 \backslash G \;\approx\; \mathbb{T}^n \times \mathbb{R}$.
Compact locally homogeneous space

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Left invariant second order equations in TM in the $(x; v) = (w, u; \xi, \eta)$ coordinates

$$\frac{dx}{dt} = xv$$
$$\frac{dv}{dt} = F(v)$$

We assume J-reversibility, where

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The *L-H system*.

$$\frac{d\xi}{dt} = \eta H(\xi), \quad \frac{d\eta}{dt} = -\langle H(\xi), \xi \rangle \qquad \text{Euler eqation}$$
$$\frac{dw}{dt} = e^{uL}\xi, \quad \frac{du}{dt} = \eta. \tag{1}$$

where $H(\xi)$ is a vector field in the unit ball of $\mathbb{R}^n$.

Any left-invariant J-reversible second order system on SM has this form.

For $H(\xi) = L^*\xi$ we obtain the geodesic flow of the left invariant metric.

For $H(\xi) = L^*\xi - C\xi$, we get the geodesic flow of the left-invariant linear connection $\widehat{\nabla} = \nabla + B$ on the group G, where for $X = \xi + \eta b, \xi \in \mathfrak{g}_0, b \perp \mathfrak{g}_0$

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Definition (1)

We call a dynamical system *semi-integrable* if an open subset U of the phase space is filled with invariant tori carrying the quasi-periodic motions. If U is also dense in the phase space then the dynamical system will be called *integrable*.

Definition (2)

For a smooth vector field $H = H(\zeta)$ defined on the closed unit ball in $\mathfrak{g}_0$, we say that a point ζ_0 in the open unit ball B is *escaping* if the integral curve $\zeta = \zeta(s)$ of H through ζ_0 is defined in a finite closed interval $[s_-, s_+] \ni 0$, $\zeta(0) = \zeta_0$, $\zeta(s) \in B$ for $s \in (s_-, s_+)$, the endpoints $\zeta(s_\pm)$ belong to the unit sphere $S = \partial B$, and the vector field H is transversal to the unit sphere S at the endpoints of the integral curve. The integral curve through an escaping point is called an *escaping trajectory*.

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For a given smooth vector field H the set of escaping points in the unit ball is open.

Any escaping trajectory of the vector field $H = H(\xi)$ gives rise to a periodic solution of the Euler equation.

Any periodic solution of the Euler equation gives rise to quasi-periodic solutions of the $L - H$ system.

This leads to

Theorem (A)

If a vector field H has escaping points in the unit ball then the $L - H$ system on SM is semi-integrable.

If the vector field H has a dense set of escaping points in the unit ball then the $L - H$ system on SM is integrable.

The tori in the $2n + 1$-dimensional phase space SM have the dimension $n + 1$.

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For quadratic $L - H$ system we can formulate an effective criterion of integrablity.

Moreover the integrablity persists under small perturbations in the space of $L - H$ systems.

Theorem (B)

If for a linear vector field $H = H\xi$ the matrix H has eigenvalues with both positive and negative real parts then the quadratic $L - H$ system on TM is integrable. Moreover the invariant tori are common level sets of real-analytic first integrals.

Any small perturbation of such a quadratic integrable $L - H$ system in the space of all $L - H$ systems must be integrable.

The last theorem covers in particular the examples of Butler (1999), and Bolsinov and Taimanov (2000).

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For a quadratic $L - H$ system there are always invariant submanifolds

$$\mathcal{A}_{\pm} = \{(w, u; \xi, \eta) \in SN| \ \ \xi = 0, \eta = \pm 1\} \approx N = \mathbb{T}^n \times \mathbb{R}/\sim$$

The dynamics on $\mathcal{A}_{\pm}$ is given by

$$\dot{w} = 0, \quad \dot{u} = \pm 1$$

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Theorem (C)

Trajectories of H contained completely in the open unit ball are covered by solutions of the $L - H$ system with positive and negative Lyapunov exponents.

This observation appears implicitely in Butler and Paternain (2008). If $div\ H = 0$ then the $L - H$ system preserves the Lebesgue measure in SN.

Theorem (D)

For a residual subset in the space of quadratic $L - H$ systems in SN there are no real-analytic invariant densities. For the open family of integrable quadratic $L - H$ systems in SN with H with all different real eigenvalues, there are C^∞ invariant densities. Moreover there is a dense subset in this family with real-analytic invariant densities.

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Butler (2003,2006) constructed C^∞ examples of hamiltonian systems with the coexistence of quasi-periodic behavior in an open subset and positive metric-entropy. In quadratic $L - H$ systems we find an open family of non-hamiltonian examples of this kind.

Theorem (E)

If H has all eigenvalues on the imaginary axis and $H \neq -H^*$ then the quadratic $L - H$ system on SN is semi-integrable, volume-preserving, and it has positive metric entropy.

Using the KAM theory one can obtain this phenomenon robustly in the class of real-analytic hamiltonian nonlinear vector fields H.

Question

Is there a hamiltonian real-analytic example?

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Using the modified linear vector field $H_1 = H - \epsilon I$, we obtain for $\epsilon > 0$ the asymptotic stability of the origin $\xi = 0$. At the same time for small ϵ the restriction of the vector field H_1 to the unit ball still has an open set of escaping trajectories.

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There is an open set of quadratic $L - H$ systems on SN which are semi-integrable and the interior of the complement to the set of invariant tori is filled with orbits which are asymptotic as $t \to +\infty$ $(-\infty)$ to the attractor $\mathcal{A}_+$ (repellor $\mathcal{A}_-$) carrying the suspension of the toral automorphism e^L (e^{-L}).

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$$\frac{d}{dt}x = v, \nabla_v v = E - \frac{\langle E, v\rangle}{\langle v, v\rangle} v, \qquad (4)$$

where $x = x(t) \in M$ is a parametrized curve in the Riemmanian manifold M and E is a tangent vector field. $k = v^2$ is a first integral. For the homogenous M and E the "vertical" left invariant vector field we get a family of $L - H$ systems, where $H(\xi) = L^*\xi - \frac{1}{k}\xi$.

Corollary

Let e^L be a hyperbolic toral automorphism and $r_{min} < 0$ and $r_{max} > 0$ be the smallest and the largest real parts of the eigenvalues of L. For $k < \frac{1}{r_{max}}$ the Gaussian thermostat (4) has a global attractor which is Anosov (the suspension of the automorphism e^L). For $k > \frac{1}{r_{max}}$ the Gaussian thermostat (4) is integrable.

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$$ds^2 = e^{-2u\lambda_1} dw_1^2 + e^{-2u\lambda_2} dw_2^2 + \cdots + e^{-2u\lambda_n} dw_n^2 + du^2.$$

Another example for $u > 0$

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where $\alpha_i, i = 1, \ldots, n$ are positive functions of the variable u alone, defined on the interval (a, b).

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If $\lim_{u\to a} \alpha_k(u) = 0$ and $\lim_{u\to b} \alpha_l(u) = 0$ for some k and l then the geodesic flow is integrable.

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$$ds^2 = e^{-2u\lambda_1} dw_1^2 + e^{-2u\lambda_2} dw_2^2 + \cdots + e^{-2u\lambda_n} dw_n^2 + du^2.$$

Another example for $u > 0$

$$ds^2 = u^{2\tau_1} dw_1^2 + u^{2\tau_2} dw_2^2 + \cdots + u^{2\tau_n} dw_n^2 + du^2$$

For $a < u < b$

$$ds^2 = \alpha_1^2 dw_1^2 + \alpha_2^2 dw_2^2 + \cdots + \alpha_n^2 dw_n^2 + du^2$$

where $\alpha_i, i = 1, \dots, n$ are positive functions of the variable u alone, defined on the interval (a, b).

Theorem (G)

If $\lim_{u\to a} \alpha_k(u) = 0$ and $\lim_{u\to b} \alpha_l(u) = 0$ for some k and l then the geodesic flow is integrable.

Proof: We pass to the hamiltonian formulation.

$$p_0 = \dot{u},\ p_j = \alpha_j^2 \dot{w}_j, j = 1, \ldots, n,$$

$$H = \frac{1}{2}\left(p_0^2 + \sum_{j=1}^{n} \alpha_j^{-2} p_j^2\right).$$

We have the obvious n first integrals $p_1, \ldots, p_n$. Together with H we get $n+1$ first integrals in involution. Compactness of the common level set? For $p_k \neq 0$ and $p_l \neq 0$

$$H \geq \alpha_l^{-2} p_i^2 + \alpha_l^{-2} p_l^2 \to +\infty \text{ as } u \to a, b.$$

hence the level set is bounded in the variable u.
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THANK YOU FOR YOUR ATTENTION!